The Annals of Applied Probability
2012, Vol. 22, No. 2, 702–733
DOI: 10.1214/11-AAP783

LIMIT THEOREMS FOR A RANDOM DIRECTED SLAB GRAPH[1]

BY D. DENISOV, S. FOSS AND T. KONSTANTOPOULOS

Cardiff University, Heriot–Watt University and Uppsala University

We consider a stochastic directed graph on the integers whereby a directed edge between i and a larger integer j exists with probability p_{j-i} depending solely on the distance between the two integers. Under broad conditions, we identify a regenerative structure that enables us to prove limit theorems for the maximal path length in a long chunk of the graph. The model is an extension of a special case of graphs studied in [*Markov Process. Related Fields* **9** (2003) 413–468]. We then consider a similar type of graph but on the "slab" $\mathbb{Z} \times I$, where I is a finite partially ordered set. We extend the techniques introduced in the first part of the paper to obtain a central limit theorem for the longest path. When I is linearly ordered, the limiting distribution can be seen to be that of the largest eigenvalue of a $|I| \times |I|$ random matrix in the Gaussian unitary ensemble (GUE).

1. Introduction. Consider a random directed graph with vertex $V = \mathbb{Z}$, the integers. A pair of integers (i, j) is declared to be an edge, directed from i to j, with probability p_{j-i} which depends only on the difference $j - i$, and this is done independently from pair to pair. We assume that $p_k = 0$ for all $k \le 0$, so there are no directed edges from a larger integer to a smaller one. We are interested in limit theorems (law of large numbers and central limit theorem) for the maximum length $T[1, n]$ of all paths from 1 to n, as $n \to \infty$. The problem as such is related to last-passage percolation.

Unlike nearest-neighbor graphs [3, 29], the quantity $T[1, n]$ does not have a direct subadditive property. It turns out that a related quantity, namely, the maximum $L[1, n]$ of all paths in the restriction of the graph on $\{1, \dots, n\}$, has an almost subadditive property [see (2)] and, thus, $L[1, n]/n \to C$, almost surely, for some deterministic constant $C \le 1$. In fact, it will be shown that there exist random vertices I, J, with $I < J$ a.s., such that, a.s., if $i < I$ and $j > J$, then vertex i is connected to vertex j by a path. From this, it follows that $T[1, n]$ has the same asymptotic properties as $L[1, n]$. The minimal condition we need to carry out our program is

$$\sum_{k=1}^{\infty} (1 - p_1) \cdots (1 - p_k) < \infty.$$

Received May 2010; revised April 2011.

[1]Supported in part by EPSRC Grant EP/E033717/1 and by the Isaac Newton Institute for Mathematical Sciences.

MSC2010 subject classifications. Primary 05C80, 60F17; secondary 60K35, 06A06.

Key words and phrases. Random graph, partial order, functional central limit theorem, GUE, last passage percolation.

Under this condition, we can identify a random subset $\mathscr{S}$ (we call it "skeleton") of $\mathbb{Z}$ whose points form a stationary renewal process (see Sections 4 and 5) over which the graph regenerates and has the property that any element v of $\mathscr{S}$ is connected by a path (directed either toward v or away from it) to any other vertex in $\mathbb{Z}$. The quantity $L[1, n]$ becomes additive over the regenerative set $\mathscr{S}$, enabling us to prove, under the stronger condition

$$\sum_{k=1}^{\infty} k(1-p_1)\cdots(1-p_k) < \infty,$$

a (functional) central limit theorem. The latter condition implies finiteness of variance of the longest path between two successive points of $\mathscr{S}$. To prove the latter assertion, we provide a rather nontrivial algorithmic construction of the last nonpositive element of $\mathscr{S}$. This construction is related to the so-called coupling-from-the past method for perfect simulation [19, 34] and is the topic of Section 7 which is based on the properties of two stopping times studied in Section 6. The central limit theorem is proved in Section 8.

We then consider an extension of the random graph on the vertex set $\mathbb{Z} \times I$, where I is a partially ordered set under some partial order $\preceq$ possessing a minimum and a maximum element. We let an edge from (x, i) to (y, j) exist with probability that depends on $y - x$ and on i and j, and only when $y - x > 0$ and $i \preceq j$. We let L_N be the length of the longest path in the restriction of the graph on $\{0, \dots, N\} \times I$ and show that the law of L_N, appropriately normalized, satisfies a functional central limit theorem such that the limit process $(Z_t, t \geq 0)$ is a $1/2$-self-similar, non-Gaussian, continuous process with Z_1 having the law of the largest eigenvalue of an a $|I| \times |I|$ random matrix in the Gaussian Unitary Ensemble (GUE) [2].

The case where all the p_k are equal to p corresponds to a directed version of the classical Erdős–Rényi graph [5]. Indeed, let $G_{n,p}$ be the Erdős–Rényi graph on the set of vertices $\{1, \dots, n\}$. To each $\{i, j\}$ which is an edge in $G_{n,p}$, we give an orientation from $i \wedge j$ to $i \vee j$. The directed graph thus obtained is precisely the restriction of our graph on the set $\{1, \dots, n\}$. This model was also studied in [18]. In this paper, among other things, sharp estimates for the $C \equiv C(p)$ as a function of p were obtained. Besides purely mathematical interest, this model is motivated by applications in Mathematical Biology (community food webs) [14, 31, 32], in Computer Science (parallel processing systems) [23] and in Physics. Allowing the connectivity probability to depend on the distance between two vertices i and j means larger modeling flexibility, on one hand, while making the model more realistic on the other.

In [18] a generalization of Borovkov's theory of renovating events [9–13] was developed in order to construct a Markov chain in infinite dimensions describing the "weights" of vertices. As a matter of fact, in [18], the random graph was a special case of a more general stochastic dynamical system (the "infinite bin model") with stationary and ergodic input. In this paper, we follow a different approach, one, that is, applicable specifically for cases where there is independence between links. In such a case, the approach has the advantage that it is more elementary using, essentially, renewal theory and coupling between renewal processes.

2. The line model. We are given a set of numbers $(p_j, j \in \mathbb{N})$, such that

$$0 \le p_j < 1, \qquad j \in \mathbb{N},$$

and consider $(\alpha_{i,j}, i, j \in \mathbb{Z}, i < j)$ as a collection of i.i.d. random variables with common law

$$\mathbb{P}(\alpha_{0,1} = 1) = 1 - \mathbb{P}(\alpha_{0,1} = -\infty) = p_{j-i}.$$

Based on this collection, we build a directed random graph G on $\mathbb{Z}$ with edges

$$E = \{(i, j) \in \mathbb{Z} \times \mathbb{Z} : i < j, \alpha_{i,j} = 1\}.$$

We shall occasionally refer to the restriction $G[i, j]$ of the graph on the vertex set $\{i, i+1, \dots, j\}$ (deleting all edges with either of the endpoints not in this set). We are interested in the behavior of the longest paths. A path π is an increasing sequence of vertices $\pi = (i_0, i_1, \dots, i_\ell)$ successively connected by edges, that is, $\alpha_{i_0,i_1} = \cdots = \alpha_{i_{\ell-1},i_\ell} = 1$. The number $\ell = |\pi|$ of edges is the length of this path. If $i_0 = i$ and $i_\ell = j$, we say that this is a path from i to j. We denote this event by $i \rightsquigarrow j$ and may also express it by saying that i leads to j or that j is reachable from i.

For any $\ell \ge 1$ and any increasing sequence $(i_0, i_1, \dots, i_\ell)$ of vertices, we conveniently define

$$|(i_0, i_1, \dots, i_\ell)| = (\alpha_{i_0,i_1} + \alpha_{i_1,i_2} + \cdots + \alpha_{i_{\ell-1},i_\ell})^+. \tag{1}$$

Clearly, this quantity is 0 if one of the summands takes value $-\infty$; otherwise, it equals ℓ. In other words, $|(i_0, i_1, \dots, i_\ell)| > 0$ if and only if $(i_0, i_1, \dots, i_\ell)$ is a path.

We let $T[i, j]$ be the maximum length of all paths from i to j. Unlike nearest-neighbor directed graph models (see, e.g., [28]), this quantity does not have a subadditivity property. To remedy this, we let $L[i, j]$ be the maximum length of all paths from some $i' \ge i$ to some $j' \le j$, that is,

$$L[i, j] = \max_{i \le i' \le j' \le j} T[i', j'].$$

That is, $L[i, j]$ is the longest path of the restricted graph $G[i, j]$. Clearly, $L[i, j]$ has the same law as $L[0, j-i]$. It is also clear that $L[i, j]$ is subadditive in the sense that

$$L[i, k] \le L[i, j] + L[j, k] + 1, \qquad i < j < k. \tag{2}$$

Indeed, if π is a path of maximal length in $G[i, k]$, then its restriction π' on $G[i, j]$ has length at most $L[i, j]$ and its restriction π'' on $G[j, k]$ has length at most

$L[j,k]$. Now the length of π is equal to the length of π' plus the length of π'' plus, possibly, 1, if j is not a vertex of π. By the subadditive ergodic theorem [26], page 192, there exists a deterministic $C \in [0,1]$ such that

$$\mathbb{P}\Big(\lim_{j\to\infty} L[i,j]/j = C\Big) = 1. \tag{3}$$

3. A stationary-ergodic framework. We now define the model on an appropriate probability space. We do this because it becomes clear what we mean by ergodicity and also because some of the results below (e.g., the existence of a skeleton—see Section 4) do not depend on the independence assumptions between the random variables $\alpha_{i,j}$. We do this as follows. To each $i \in \mathbb{Z}$ we assign a vector $\boldsymbol{\delta}^{(i)} = (\delta_j^{(i)}, j \in \mathbb{Z})$ with $\{-\infty, 1\}$-valued components. Let Ω contain elements of the form

$$\omega = \big(\boldsymbol{\delta}^{(i)}, i \in \mathbb{Z}\big),$$

where

$$\boldsymbol{\delta}^{(i)} = \big(\delta_j^{(i)}, j \in \mathbb{Z}\big)$$

consisting of $\delta_j^{(i)} \in \{-\infty, 1\}$. So $\Omega = (\{-\infty,1\}^{\mathbb{Z}})^{\mathbb{Z}} = \{-\infty,1\}^{\mathbb{Z}\times\mathbb{Z}}$, and is equipped with the product sigma-field. A natural shift θ on Ω is the map defined by

$$\omega = \big(i \mapsto \boldsymbol{\delta}^{(i)}\big) \mapsto \theta\omega = \big(i \mapsto \boldsymbol{\delta}^{(i+1)}\big). \tag{4}$$

Assume that we equip Ω by a probability measure $\mathbb{P}$ which preserves θ, namely, $\mathbb{P}(\theta A) = \mathbb{P}(A)$ for all measurable subsets A of Ω, and that θ is ergodic. We also assume that

$$\mathbb{P}\big(\boldsymbol{\delta}_k^{(0)} = -\infty\big) = 1, \qquad k \le 0.$$

We define random variables $\alpha_{i,j}$ by

$$\alpha_{i,j}(\omega) = \delta_{j-i}^{(i)}.$$

Hence,

$$\alpha_{i,j}(\theta\omega) = \delta_{j-i}^{(i+1)} = \delta_{(j+1)-(i+1)}^{(i+1)} = \alpha_{i+1,j+1}(\omega).$$

The object of interest is the directed graph $G(\omega) = (\mathbb{Z}, E(\omega))$ with $(i,j) \in E(\omega)$ iff $\alpha_{i,j}(\omega) = 1$.

The random variables $L[i,j]$ are all defined explicitly on Ω via

$$L[i,j] = \max_{i \le i_0 < i_1 < \cdots < i_\ell \le j} |(i_0, \dots, i_\ell)|,$$

where $|(i_0, \dots, i_\ell)|$ is the random variable defined by (1). The subadditive ergodic theorem holds under this general framework and so (3) is valid.

A word on notation: If $(A_n, n \in \mathbb{Z})$ is a collection of events of Ω and τ is a $\mathbb{Z}$-valued random variable on Ω, then A_τ denotes the event containing all $\omega \in \Omega$ such that $\omega \in A_{\tau(\omega)}$.

4. The skeleton. The framework here is that of the previous section. Recall the shorthand $\{i \rightsquigarrow j\} = \{T[i,j] > 0\}$ for the event that there is a path from i to j. Consider, for each $n \in \mathbb{Z}$, the events

$$A_n^+ := \bigcap_{j>n} \{n \rightsquigarrow j\} = \{\text{any } j > n \text{ is reachable from } n\},$$

$$A_n^- := \bigcap_{j<n} \{j \rightsquigarrow n\} = \{n \text{ is reachable from any } j < n\}.$$

We are interested in the random set

$$\mathscr{S}(\omega) := \{n \in \mathbb{Z} : \omega \in A_n^+ \cap A_n^-\}, \tag{5}$$

and refer to it as the *skeleton* of the random graph. The terminology is supposed to be reminiscent of a point of view described next.

Let $\mathscr{P}(E) \subset \mathbb{Z} \times \mathbb{Z}$ be a partial order [i.e., if $(i,j) \in \mathscr{P}(E)$ and $(j,k) \in \mathscr{P}(E)$, then $(i,k) \in \mathscr{P}(E)$] which contains the set of edges E. In fact, take $\mathscr{P}(E)$ to be the smallest such set. Necessarily, $\mathscr{P}(E) = \{(i,j) \in \mathbb{Z} \times \mathbb{Z} : i \rightsquigarrow j\}$. A subset U of $\mathbb{Z}$ is totally ordered under the partial order $\rightsquigarrow$ if for any distinct $i, j \in U$ we either have $i \rightsquigarrow j$ or $j \rightsquigarrow i$. We say that a totally ordered subset U is *special* if it has the stronger property that for all distinct i, j with $i \in U$ and $j \in V$, we either have $i \rightsquigarrow j$ or $j \rightsquigarrow i$. Clearly, the union of special totally ordered subsets is special and totally ordered; thus, we can speak of the maximal special totally ordered subset; we refer to it as the skeleton of the partial order. Adopting this definition, it is now clear that the set $\mathscr{S}$ defined by (5) is the skeleton of the partial order $\rightsquigarrow$ on $\mathbb{Z}$. In [1] the elements of $\mathscr{S}$ are referred to as *posts*. In fact, [1] uses $\mathscr{S}$ in order to derive limit theorems of the number N_n of linear extensions of the random partial order $\rightsquigarrow$ on $\{1, \dots, n\}$.

For a general partially ordered set, a skeleton may not exist. However, in our case, the condition $\mathbb{P}(A_0^+ \cap A_0^-) > 0$ is sufficient for $\mathscr{S}$ to be almost surely infinite.

LEMMA 1. *If* $\lambda := \mathbb{P}(A_0^+ \cap A_0 -) > 0$, *then* $\mathscr{S}$ *is an a.s. infinite set.*

PROOF. Let θ be the shift defined by (4). Then, for all ω, $\mathscr{S}(\omega) = \mathscr{S}(\theta\omega)$. Since $\mathbb{P}$ is θ-invariant, the result follows. $\square$

Assuming that $\lambda = \mathbb{P}(A_0^+ \cap A_0^-) > 0$, we may then, equivalently, consider $\mathscr{S}$ as a stationary-ergodic point process on the integers with rate λ because $\lambda = \mathbb{P}(0 \in \mathscr{S})$. We let Γ_n, $n \in \mathbb{Z}$, be an enumeration of the elements of $\mathscr{S}$ according to the following convention:

$$\cdots < \Gamma_{-1} < \Gamma_0 \le 0 < \Gamma_1 < \Gamma_2 < \cdots.$$

In particular, Γ_0 is the largest nonpositive element of $\mathscr{S}$.

We can now strengthen the subadditivity property (2) for L:

LEMMA 2. *For all integers $m < n$,*

$$L[\Gamma_m, \Gamma_n] = L[\Gamma_m, \Gamma_{m+1}] + \cdots + L[\Gamma_{n-1}, \Gamma_n].$$

PROOF. To see this, consider the interval $[\Gamma_1, \Gamma_n]$ and a path π^* of length $L[\Gamma_1, \Gamma_n]$. Then this path must visit all the intermediate skeleton points $\Gamma_1, \dots, \Gamma_n$. Indeed, suppose this is not the case and π^* does not visit, say, Γ_l, for some $1 \le l \le n$. Consider an edge (i, j) belonging to π^*, with $i \le \Gamma_l \le j$. By the definition of Γ_l, both (i, Γ_l) and (Γ_l, j) are edges of the random graph G. Therefore, we can increase the length of π^* by 1 if we replace the edge (i, j) by two edges (i, Γ_l) and (Γ_l, j). This leads to contradiction since π^* has length $L[\Gamma_1, \Gamma_n]$ which is, by definition, maximal. □

5. Regenerative structure. We shall henceforth specialize to the i.i.d. case. Specifically, assume that

$$\mathbb{P}(\delta_j^{(0)} = 1) = \begin{cases} 0, & \text{if } j \le 0, \\ p_j, & \text{if } j > 0, \end{cases}$$

and that $(\boldsymbol{\delta}^{(i)}, i \in \mathbb{Z})$ are i.i.d.

Throughout, we make use of the following two conditions:

$$\text{(C1)} \quad 0 < p_1 < 1,$$

$$\text{(C2)} \quad \sum_{k=1}^{\infty} (1 - p_1) \cdots (1 - p_k) < \infty.$$

We also sometimes write $q_j = 1 - p_j$. For each $j \in \mathbb{Z}$ we consider its immediate neighbors:

$$\begin{aligned} \overline{\eta}(j) &:= \min\{k > j : \alpha_{j,k} = 1\}, \\ \overline{\xi}(j) &:= \max\{i < j : \alpha_{i,j} = 1\}. \end{aligned} \tag{6}$$

See Figure 1. The distances of these vertices from j are denoted as follows:

$$\begin{aligned} \eta(j) &:= \overline{\eta}(j) - j, \\ \xi(j) &:= j - \overline{\xi}(j). \end{aligned}$$

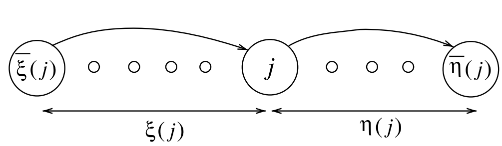

FIG. 1. *Notation used: $\overline{\xi}(j)$ is the first vertex below j, that is, connected to j; correspondingly, $\overline{\eta}(j)$ is the first vertex above j connected to j.*

Notice that $(\xi(j), j \in \mathbb{Z})$ and $(\eta(j), j \in \mathbb{Z})$ are identically distributed sequences, and that each one is a sequence of i.i.d. random variables. Furthermore, for each $j \in \mathbb{Z}$,

$$\big(\xi(j), \xi(j-1), \dots\big) \perp\!\!\!\perp \big(\eta(j), \eta(j+1), \dots\big).$$

Henceforth, we shall let ξ be a random variable with distribution the common distribution of $\xi(j)$ and $\eta(j)$:

$$\mathbb{P}(\xi > n) = \mathbb{P}\big(\xi(0) > n\big) = \mathbb{P}\big(\eta(0) > n\big) = (1 - p_1) \cdots (1 - p_n), \qquad n \in \mathbb{N}.$$

For integers $u < v$, define the events

$$A^+_{u,v} := \bigcap_{j=u+1}^{v} \{u \rightsquigarrow j\}, \qquad A^-_{u,v} := \bigcap_{j=u}^{v-1} \{j \rightsquigarrow v\}, \tag{7}$$

for which, clearly,

$$A^+_{u,v} \supset A^+_{u,v+1}, \qquad A^-_{u,v} \supset A^-_{u-1,v},$$

with

$$\lim_{v\to\infty} A^+_{u,v} = A^+_u, \qquad \lim_{u\to-\infty} A^-_{u,v} = A^-_v. \tag{8}$$

Furthermore,

$$A^+_{u,v} \cap A^+_{v,w} \subset A^+_{u,w} \qquad \text{if } u < v < w, \tag{9}$$

a property we shall use in Section 7. Observe also the following:

LEMMA 3. *For all integers $u < v$,*

$$A^+_{u,v} = \bigcap_{j=u+1}^{v} \bigcup_{i=u}^{j-1} \{i \rightsquigarrow j\} = \bigcap_{j=u+1}^{v} \{u \le \overline{\xi}(j)\},$$

$$A^-_{u,v} = \bigcap_{j=u}^{v-1} \bigcup_{i=j+1}^{v} \{j \rightsquigarrow i\} = \bigcap_{j=u}^{v-1} \{\overline{\eta}(j) \le v\},$$

$$A^+_u = \bigcap_{j>u} \bigcup_{i=u}^{j-1} \{i \rightsquigarrow j\} = \bigcap_{j>u} \{u \le \overline{\xi}(j)\},$$

$$A^-_v = \bigcap_{j<v} \bigcup_{i=j+1}^{v} \{j \rightsquigarrow i\} = \bigcap_{j<v} \{\overline{\eta}(j) \le v\}.$$

PROOF. We prove the first equality. That $A^+_{u,v} \subset \bigcap_{j=u+1}^{v} \bigcup_{i=u}^{j-1} \{i \rightsquigarrow j\}$ is immediate from the definition (7). To prove the opposite inclusion, assume that $v >$

$u+1$ (otherwise there is nothing to prove) and that for all integers $j \in [u+1, v]$ there exists an integer $i \in [u, j-1]$ such that $i \rightsquigarrow j$. Fix $j > u$ and pick i_1 to be the largest among the vertices between u and $j-1$ such that $i_1 \rightsquigarrow j$; necessarily, $\alpha_{i_1,j} = 1$. Then pick the largest vertex i_2 among the vertices between u and $i_1 - 1$ such that $i_2 \rightsquigarrow i_1$, and continue this way. Since $i_1 > i_2 > \cdots \geq u$, it follows that this process terminates with some $i_k = u$. Since $(u = i_k, i_{k+1}, \ldots, i_1, j)$ is a path, we have that $u \rightsquigarrow j$. The second equality for $A^+_{u,v}$ now follows from the definition (6). The relations for $A^-_{u,v}$ follow similarly. The third (resp., fourth) line is obtained by sending v to $+\infty$ (resp., u to $-\infty$) in the first (resp., second) one. $\square$

This lemma tells us that $A^+_{u,v}$ is the intersection of $v-u$ independent events. Indeed, since $\overline{\xi}(j) = j - \xi(j)$, we have

$$A^+_{u,v} = \{\xi(u+1) \leq 1, \xi(u+2) \leq 2, \ldots, \xi(v) \leq v-u\}, \tag{10}$$

and the random variables $\xi(u+1), \ldots, \xi(v)$ are i.i.d. Similarly, for $A^-_{u,v}$,

$$A^-_{u,v} = \{\eta(u) \leq v-u, \ldots, \eta(v-2) \leq 2, \eta(v-1) \leq 1\}. \tag{11}$$

Moreover, since

$$\big(\xi(u+1), \xi(u+2), \ldots, \xi(v)\big) \stackrel{d}{=} \big(\eta(v-1), \eta(v-2), \ldots, \eta(u)\big),$$

we have that $\mathbb{P}(A^+_{u,v}) = \mathbb{P}(A^-_{u,v})$. Similarly, both A^+_n and A^-_n are intersections of infinitely many independent events:

$$A^+_n = \bigcap_{j>n} \{\xi(j) \leq j-n\}, \tag{12}$$

$$A^-_n = \bigcap_{j<n} \{\eta(j) \leq n-j\}, \tag{13}$$

and $\mathbb{P}(A^+_n) = \mathbb{P}(A^-_n)$. The skeleton (5) can be expressed as follows:

$$\mathscr{S} = \Big\{n \in \mathbb{Z} : \sup_{i<n} \overline{\eta}(i) \leq n \leq \inf_{j>n} \overline{\xi}(j)\Big\}. \tag{14}$$

Regarding $\mathscr{S}$ as a point process, we see that it has rate

$$\lambda = \mathbb{P}(0 \in \mathscr{S}) = \mathbb{P}(A^+_0)^2 = \left(\prod_{j=1}^{\infty} \mathbb{P}\big(\xi(j) \leq j\big)\right)^2 = \prod_{j=1}^{\infty} \big[1 - \mathbb{P}\big(\xi(0) > j\big)\big]^2.$$

Since

$$\mathbb{P}\big(\xi(0) > j\big) = \mathbb{P}(\alpha_{0,1} = \cdots = \alpha_{0,j} = 0) = (1-p_1)\cdots(1-p_j), \tag{15}$$

we have

$$\lambda = \prod_{j=1}^{\infty} [1 - (1-p_1)\cdots(1-p_j)]^2 \tag{16}$$

and so

$$(\mathrm{C2}) \iff \lambda > 0 \iff \mathbb{E}[\xi(0)] < \infty.$$

Consider now two successive skeleton points Γ_k and Γ_{k+1} and let $\mathscr{C}_k(\omega)$ be the restriction of ω on $[\Gamma_k, \Gamma_{k+1})$:

$$\mathscr{C}_k := \big(\boldsymbol{\delta}^{(n)}, \Gamma_k \le n < \Gamma_{k+1}\big), \qquad k \in \mathbb{Z};$$

we refer to it as the kth "cycle." We next show that the sequence of cycles has a regenerative structure in the following sense:

LEMMA 4. *The cycles* $(\mathscr{C}_k, k \in \mathbb{Z})$ *are independent and* $(\mathscr{C}_k, k \in \mathbb{Z} - \{0\})$ *are identically distributed.*

Intuitively, Lemma 4 is based on the following observation. Suppose that 0 is a skeleton vertex (i.e., condition on the event $A_0^- \cap A_0^+$). Then $\overline{\xi}(1) \ge 0$, $\overline{\xi}(2) \ge 0$, etc. In other words, $\overline{\xi}(1) = 0$, $\overline{\xi}(2) \in \{1, 2\}$, $\overline{\xi}(3) \in \{0, 1, 2\}$, etc. To determine the location of the next skeleton vertex after 0, we need to find the first vertex $j > 0$ which is connected with every vertex between 0 and $j - 1$. This means that, conditional on 0 being a skeleton vertex, the location of the first skeleton vertex larger than 0 does not depend on the $(\boldsymbol{\delta}^{(n)}, n < 0)$.

PROOF OF LEMMA 4. We show that, for all $j \in \mathbb{Z}$, the cycle $\mathscr{C}_j$ is independent of $(\mathscr{C}_i, \dots, \mathscr{C}_{j-1})$ for any $i < j$. By stationarity, it suffices to show that, conditional on $\{\Gamma_0 = 0\}$, the cycle $\mathscr{C}_0$ is independent of $(\mathscr{C}_{-i}, \dots, \mathscr{C}_{-1})$ for any $i > 0$. Fix integers

$$\gamma_{-i} < \cdots < \gamma_{-1} < \gamma_0 = 0 < \gamma_1.$$

Pick events

$$B_k \in \sigma\big(\boldsymbol{\delta}^{(n)}, \gamma_k \le n < \gamma_{k+1}\big)$$

and let

$$\begin{aligned} C_0 &:= \{\Gamma_1 = \gamma_1\} \cap B_0, \\ C_{-j} &:= \{\Gamma_{-j} = \gamma_{-j}\} \cap B_{-j}, \qquad j = 1, \dots, i. \end{aligned}$$

We will show that

$$\mathbb{P}(C_{-i}, \dots, C_{-1}; C_0 | \Gamma_0 = 0) = \mathbb{P}(C_{-i}, \dots, C_{-1} | \Gamma_0 = 0)\mathbb{P}(C_0 | \Gamma_0 = 0).$$

Assume that $\Gamma_0 = 0$ (i.e., 0 is a skeleton vertex). Then, by (14),

$$\dots, \overline{\eta}(-2), \overline{\eta}(-1) \le 0 \le \overline{\xi}(1), \overline{\xi}(2), \dots. \tag{17}$$

In view of the latter inequality, we have

$$\begin{aligned}\Gamma_{-1} &= \max\{n < 0 : \mathbf{1}_{A_n^- \cap A_n^+} = 1\}\\ &= \max\Big\{n < 0 : \sup_{i<n} \overline{\eta}(i) \le n \le \inf_{i>n} \overline{\xi}(i)\Big\}\\ &= \max\Big\{n < 0 : \sup_{i<n} \overline{\eta}(i) \le n \le \max_{n<i\le 0} \overline{\xi}(i)\Big\} =: \widehat{\Gamma}_{-1},\end{aligned}$$

where the last serves as a definition of a new random variable $\widehat{\Gamma}_{-1}$. Observe that $\widehat{\Gamma}_{-1}$ is measurable with respect to $\mathscr{F}^- := \sigma(\boldsymbol{\delta}^{(n)}, n < 0)$. Similarly, for $k < 0$, on the event $\Gamma_k = \gamma_k$, the random variable Γ_{k+1} is equal to some $\mathscr{F}^-$-measurable random variable $\widehat{\Gamma}_{k+1}$. We also define

$$\widehat{\Gamma}_1 := \min\{n > 0 : \overline{\eta}(0), \dots, \overline{\eta}(n-1) \le n \le \overline{\xi}(n+1), \overline{\xi}(n+2), \dots\},$$

and observe that $\widehat{\Gamma}_1$ is measurable with respect to $\mathscr{F}^+ := \sigma(\boldsymbol{\delta}^{(n)}, n > 0)$ and that, on the event $\{\Gamma_0 = 0\}$, the random variables Γ_1 and $\widehat{\Gamma}_1$ coincide. These observations and the facts that $\mathscr{F}^+$, $\mathscr{F}^-$ are independent, $\{\Gamma_0 = 0\} = A_0^- \cap A_0^+$, and $A_0^\pm \in \mathscr{F}^\pm$ justify the following:

$$\begin{aligned}\mathbb{P}(C_{-i}, \dots, C_{-1}; C_0 | \Gamma_0 = 0) &= \frac{\mathbb{P}(C_{-i}, \dots, C_{-1}, C_0, A_0^-, A_0^+)}{\mathbb{P}(A_0^- A_0^+)}\\ &= \frac{\mathbb{P}(C_{-i}, \dots, C_{-1}, A_0^-)}{\mathbb{P}(A_0^-)} \frac{\mathbb{P}(C_0, A_0^+)}{\mathbb{P}(A_0^+)}\\ &= \frac{\mathbb{P}(C_{-i}, \dots, C_{-1}, A_0^-, A_0^+)}{\mathbb{P}(A_0^-, A_0^+)} \frac{\mathbb{P}(C_0, A_0^+, A_0^-)}{\mathbb{P}(A_0^+, A_0^-)}\\ &= \mathbb{P}(C_{-i}, \dots, C_{-1} \mid \Gamma_0 = 0)\mathbb{P}(C_0 \mid \Gamma_0 = 0),\end{aligned}$$

as needed. □

COROLLARY 1. *The bivariate random variables*

$$(\Gamma_j - \Gamma_0, L[\Gamma_0, \Gamma_j]), \qquad j \ge 1; \qquad (\Gamma_0 - \Gamma_{-j}, L[\Gamma_0, \Gamma_{-j}]), \qquad j \ge 1,$$

are i.i.d. and independent of $(\Gamma_1 - \Gamma_0, L[\Gamma_1, \Gamma_0])$.

We note that the set $\mathscr{S}$ with elements $(\Gamma_k, k \in \mathbb{Z})$ forms a stationary renewal process. That it is stationary is clear from the general setup.

6. Two stopping times. In this section we study properties of the following two random variables:

$$\begin{aligned}\mu &:= \inf\{i > 0 : \mathbf{1}_{A_{-i,0}^-} = 0\},\\ \nu &:= \inf\{i > 0 : \mathbf{1}_{A_{-i,0}^+} = 1\}.\end{aligned}$$

These random variables are important in the algorithmic construction of Section 7.

Note that $-\nu$ is the first vertex < 0 with the property that every vertex in the interval $(-\nu, 0]$ is reachable from $-\nu$:

$$\nu = \inf\{i > 0 : -\nu \rightsquigarrow 0, -\nu \rightsquigarrow -1, \dots, -\nu \rightsquigarrow -\nu + 1\}.$$

Also, $-\mu$ is the first vertex < 0 such that 0 is not reachable from $-\mu$:

$$\mu = \inf\{i > 0 : -i \not\rightsquigarrow 0\}.$$

We will show that μ is a defective random variable, that is, that $\mathbb{P}(\mu = \infty) > 0$, with conditional tail $\mathbb{P}(\mu > n | \mu < \infty)$ comparable to the integrated tail of ξ. We will also show that ν is an a.s. finite random variable with the same number of moments as ξ.

Note first that both μ and ν are stopping times with respect to the filtration $(\mathscr{F}_k^-, k \le 0)$. Observe that

$$\{\mu = \infty\} = \bigcap_{i \ge 1} A^-_{-i,0} = A^-_0. \tag{18}$$

Since condition (C2) is equivalent to $\mathbb{P}(A^-_0) > 0$, we have

$$\mathbb{P}(\mu = \infty) > 0.$$

On the other hand,

$$\{\nu = \infty\} = \bigcap_{n=1}^{\infty} (A^+_{-n,0})^c,$$

and, as we shall see below, this event has probability zero:

$$\mathbb{P}(\nu = \infty) = 0. \tag{19}$$

Let us first focus on the law of μ, conditional on $\{\mu < \infty\}$. This can be computed easily, from the definition of μ, and equations (11), (18) and (15):

$$\begin{aligned}
\mathbb{P}(n < \mu < \infty) &= \mathbb{P}\big(\eta(-k) \le k \text{ for all } 1 \le k \le n\big)\mathbb{P}\big(\eta(-m) > m \text{ for some } m > n\big) \\
&= \prod_{k=1}^{n} \mathbb{P}\big(\eta(-k) \le k\big)\left(1 - \prod_{m=n+1}^{\infty} \mathbb{P}\big(\eta(-m) \le m\big)\right) \\
&= (1 - q_1)(1 - q_1 q_2) \cdots (1 - q_1 q_2 \cdots q_n) \\
&\quad \times \left(1 - \prod_{m=n+1}^{\infty} (1 - q_1 q_2 \cdots q_m)\right).
\end{aligned} \tag{20}$$

Conditional on $\{\mu < \infty\}$, the random variable μ has a tail comparable to the integrated tail of ξ:

LEMMA 5. *Suppose that* (C1) *and* (C2) *hold. There exist constants* $0 < C_1 < C_2 < \infty$ *such that, for all* $n \geq 0$,

$$C_1 \sum_{m>n}^{\infty} \mathbb{P}(\xi > m) \leq \mathbb{P}(\mu > n | \mu < \infty) \leq C_2 \sum_{m>n}^{\infty} \mathbb{P}(\xi > m).$$

PROOF. Since $p_1 < 1$, we have $\lambda < 1$ [see (16)] and so

$$\mathbb{P}(\mu < \infty) = 1 - \lambda^{1/2} > 0.$$

Using (20), we have

$$\mathbb{P}(\mu > n | \mu < \infty) \leq \frac{1}{1 - \lambda^{1/2}} \sum_{m=n+1}^{\infty} \mathbb{P}\big(\eta(m) > m\big) = \frac{1}{1 - \lambda^{1/2}} \sum_{m=n+1}^{\infty} \mathbb{P}(\xi > m).$$

Hence, $C_1 = 1/(1 - \lambda^{1/2})$. To obtain a bound from below, note that $\prod_{k=1}^{n} \mathbb{P}(\eta(-k) \leq k) \geq \mathbb{P}(\mu = \infty)$ and so (20) gives

$$\begin{aligned}
\mathbb{P}(n < \mu < \infty) &\geq \lambda^{1/2} \left(1 - \prod_{m=n+1}^{\infty} \mathbb{P}\big(\eta(-m) \leq m\big)\right) \\
&\geq \lambda^{1/2} \left(1 - \exp\left(- \sum_{m=n+1}^{\infty} \mathbb{P}(\xi > m)\right)\right) \\
&\geq \lambda^{1/2} g(\mathbb{E}\xi) \sum_{m=n+1}^{\infty} \mathbb{P}(\xi > m),
\end{aligned}$$

where $g(x) = (1 - e^{-x})/x$. Hence, $C_2 = g(\mathbb{E}\xi)\lambda^{1/2}/(1 - \lambda^{1/2})$. $\square$

We next prove something stronger than (19), namely, that ν has the same number of moments as ξ.

LEMMA 6. *If* $\mathbb{E}\xi^r < \infty$ *for some* $r \geq 1$, *then* $\mathbb{E}\nu^r < \infty$.

PROOF. By the definition of ν and equation (10), we have

$$\nu = \inf\big\{n \geq 1 : \xi(0) \leq n, \xi(-1) \leq n - 1, \ldots, \xi\big(-(n-1)\big) \leq 1\big\}.$$

Define a sequence of nonnegative random variables $x_0, x_1, x_2, \ldots$ by $x_0 = 0$ and

$$x_n = \max\big\{\xi(0) - n, \xi(-1) - (n-1), \ldots, \xi\big(-(n-1)\big) - 1\big\}, \qquad n \geq 1.$$

Then

$$\nu = \inf\{n \geq 1 : x_n = 0\}.$$

The x_n satisfy

$$x_{n+1} = \max(x_n, \xi(-n)) - 1, \qquad n \geq 0,$$

and, since the $\xi(-n)$ are i.i.d., $(x_n, n \ge 0)$ is a Markov chain in $\mathbb{Z}_+$. We now make two observations that imply the statement of the lemma. First, if $x_n > K > 0$, then

$$x_{n+1} - x_n = \bigl(\xi(-n) - x_n\bigr)^+ - 1 \le \bigl(\xi(-n) - K\bigr)^+ - 1.$$

But $\mathbb{E}[(\xi - K)^+] < 1$ for sufficiently large K. Therefore, after the Markov chain leaves the interval $[0, K]$ (for sufficiently large K), it is majorized from above by a random walk with increments distributed like $(\xi - K)^+ - 1$ whose mean is negative. By standard properties of random walks, this implies that the return time T_K to the set $[0, K]$ satisfies $\mathbb{E}T_K^r < \infty$ if $\mathbb{E}((\xi - K)^+ - 1)^r < \infty$; and the latter is equivalent to $\mathbb{E}\xi^r < \infty$. The second observation is that the Markov chain (x_n) returning to the set $[0, K]$ eventually hits point 0 after a geometric number of trials. $\square$

COROLLARY 2. *If* (C2) *holds*, *then* $\mathbb{E}\nu < \infty$.

7. Algorithmic construction of $\boldsymbol{\Gamma_0}$. In this section we give a method for constructing a specific skeleton point, for example, the first one which is to the left of the origin. This is the point Γ_0. Besides the theoretical interest, such a construction will be used later for proving a central limit theorem; it can also be used in connection to a perfect simulation algorithm for estimating the value of $C = \lim_{n\to\infty} L[1, n]/n$ (see remarks at the end of the section).

The idea for the construction of Γ_0 is this: recall that $-\nu$ is the first vertex < 0 which is connected to every point between $-\nu$ and 0. We check whether $-\nu$ is also reachable from every point from the left. If it is, we declare that $-\nu$ is a *silver* point and stop the procedure. If not, there is a first vertex before $-\nu$ which fails to be connected to $-\nu$. Using the shift operator θ defined in (4), this vertex is at distance $\mu \circ \theta^{-\nu}$ from $-\nu$; in other words, this distance is the functional μ applied to the shifted ω, when the origin is placed at $-\nu$. We then set $\mu[1] = \nu + \mu \circ \theta^{-\nu}$, which is the location of the previous vertex, and $\nu[1] = \nu$ and this finishes the first step of the procedure.

The second step of the algorithm is similar to the first one: we search for the first vertex $-\nu[2]$ before $-\mu[1]$ which is connected to every vertex between $-\nu[2]$ and $-\nu[1]$. We know that we can find such a vertex with probability one. If it also happens that $-\nu[2]$ is reachable from any point from the left, we stop and declare $-\nu[2]$ as our *silver* point. Otherwise, there will be a first vertex, $-\mu[2] < -\nu[2]$, which fails to be connected to $-\nu[2]$.

The procedure continues in the same way, until the first *silver* point is found, and it will be found with probability one. This first silver point will have the property that it is reachable from every point from the left and is connected to every point up until the origin; see Lemma 9 below. The distribution of this first silver point is well understood and this is the content of Lemma 8. In fact, we will show that there are infinitely many silver points which form a (delayed) renewal process backward;

see Lemma 11. Finally, in Theorem 1 we show that among the infinitude of silver points we can pick a *gold* one, namely, the point Γ_0.

To define the algorithm explicitly, we consider a sequence of $\mathbb{N} \cup \{+\infty\}$-valued stopping times relative to the filtration $(\mathscr{F}_k, k \geq 1)$, defined as follows. Let

$$\begin{aligned} &\nu[1] := \nu, \\ &\mu[1] := \nu + \mu \circ \theta^{-\nu} = \inf\{j > \nu : \mathbf{1}_{A^-_{-j,-\nu}} = 0\}, \end{aligned} \tag{21}$$

and, recursively, for $k \geq 2$,

$$\begin{aligned} &\nu[k] := \inf\{j > \mu[k-1] : \mathbf{1}_{A^+_{-j,-\nu[k-1]}} = 1\}, \\ &\mu[k] := \nu[k] + \mu \circ \theta^{-\nu[k]} = \inf\{j > \nu[k] : \mathbf{1}_{A^-_{-j,-\nu[k]}} = 0\}, \end{aligned} \tag{22}$$

where θ is the natural shift (4). It is understood that if for some k we have $\mu[k] = \infty$, then $\nu[j] = \mu[j] = \infty$ for all $j \geq k+1$. We thus obtain an increasing sequence of stopping times

$$\nu = \nu[1] < \mu[1] < \nu[2] < \mu[2] < \nu[3] < \mu[3] < \cdots$$

which [since $\mathbb{P}(\mu = \infty) > 0$] is eventually equal to infinity. It is convenient to think of these stopping times as the points of an alternating point process (the μ-points and the ν-points). In words, the sequence of these stopping times is defined by first laying a ν-point in location $\nu[1]$. Then, as long as $\eta(-(\nu[1]+i)) \leq i$ for $i = 1, 2, \dots,$ we place no point in location $\nu[1]+i$. At the first instance i at which $\eta(-(\nu[1]+i)) > i$, we place a μ-point in location $\nu[1]+i$ and call it $\mu[1]$. The random variables $(\eta(-(\nu[1]+i)), i \geq 1)$ are independent of $\nu[1]$, and so the event that we place a μ-point in a finite location is independent of $\nu[1]$ and has probability $\mathbb{P}(\mu < \infty) = 1 - \lambda^{1/2}$. The procedure continues in the same way: having placed $\nu[k] < \infty$, we decide, independently of the past (i.e., $\mathscr{F}^-_{\nu[k]}$), whether to create a new μ-point or not (i.e., place it at infinity). If we do create a new μ-point $\mu[k]$, then, clearly, $\nu[k+1]$ is also finite and $\nu[k+1] - \nu[k]$ has the same distribution as $\nu[2] - \nu[1]$ conditional on $\mu[1] < \infty$. Thus, for each ω, the recursion stops at the index

$$K := \inf\{k \geq 1 : \mu[k] = \infty\}. \tag{23}$$

From the discussion above we immediately obtain the following:

LEMMA 7. *Assume that* (C1) *and* (C2) *hold. Then K is a geometric random variable with*

$$\mathbb{P}(K > k) = (1 - \lambda^{1/2})^k, \qquad k \geq 0.$$

By definition, $\mu[K] = \infty$ but $\mu[K-1] < \infty$. Hence,

$$\nu[K] < \infty \qquad \text{a.s.}$$

NOTE 1. We stop for a minute to point out that the whole purpose of the construction of these random variables is the random variable $\nu[K]$. In other words, for each $\omega \in \Omega$, we apply recursion (21)–(22) to obtain the alternating sequence of ν- and μ-points, through them we define the index K as in (23) and, finally, $\nu[K]$. Thus, $\nu[K]$ is a well-defined (measurable) function of ω. We refer to $-\nu[K]$ as the first *silver* point before 0.

Although K depends on the whole alternating process $(\nu[k], \mu[k]), k \geq 1)$, we can identify the law of $\nu[K]$ as follows:

LEMMA 8. *On a new probability space, let $K, \psi_1, \psi_2, \psi_3, \dots$ be independent random variables with distributions*

$$\begin{aligned} P(K > k) &= (1 - \lambda^{1/2})^k, \qquad k \geq 0, \\ \psi_1 &\stackrel{d}{=} \nu, \\ \psi_i &\stackrel{d}{=} (\nu[2] - \nu[1] \mid \mu[1] < \infty) \\ &\stackrel{d}{=} (\inf\{j > \mu : \mathbf{1}_{A^+_{-j,0}} = 1\} \mid \mu < \infty), \qquad i \geq 2. \end{aligned}$$

Then, assuming (C1) *and* (C2),

$$\nu[K] \stackrel{d}{=} \psi_1 + \sum_{i=1}^{K-1} \psi_{i+1}. \tag{24}$$

PROOF. It follows from

$$\nu[K] = \nu[1] + \sum_{i=1}^{K-1} (\nu[i+1] - \nu[i]),$$

using a simple probabilistic argument as described above. $\square$

The reason we are interested in the random variable $\nu[K]$ is the following:

LEMMA 9. *Assume* (C1) *and* (C2) *hold. Then for $\mathbb{P}$-a.e. ω*

$$\omega \in A^-_{-\nu[K]} \cap A^+_{-\nu[K],0}. \tag{25}$$

Note that replacing the index n in a sequence of events A_n by a random index N amounts to defining the event $A_N = \{\omega \in \Omega : \text{there exists } n \text{ such that } n = N(\omega) \text{ and } \omega \in A_n\}$.

The meaning of (25) is that the vertex $\nu[K]$ of the random graph has the property that there is a path from every $j < \nu[K]$ to $\nu[K]$ and there is a path from $\nu[K]$ to every i such that $\nu[K] < i \leq 0$. Our goal is to identify a skeleton point. Whereas $\nu[K]$ is not a skeleton point for sure, there is a positive probability that it is.

PROOF OF LEMMA 9. If $K = k$, for some $k \geq 1$, then $\mu[k] = \infty$ but $\mu[k-1] < \infty$, so $\nu[k] < \infty$ and $\mathbf{1}_{A^-_{-j,-\nu[k]}} = 0$ for all $j > \nu[k]$. Hence,

$$\{K = k\} \subset \bigcap_{j > \nu[k]} A^-_{-j,-\nu[k]} = A^-_{-\nu[k]},$$

by (8). Also, if $K = k$, then $\nu[k], \nu[k-1], \dots, \nu[1] < \infty$ and so

$$\{K = k\} \subset A^+_{-\nu[k],-\nu[k-1]} \cap A^+_{-\nu[k-1],-\nu[k-2]} \cap \cdots \cap A^+_{-\nu[1],0} \subset A^+_{-\nu[k],0},$$

by (9). But K is a geometric random variable and, hence, $K < \infty$, a.s. $\square$

We also have the following result concerning moments of $\nu[K]$:

LEMMA 10. *Assume* (C1) *and* (C2) *hold. If, in addition, there exists* $r \geq 1$ *such that* $\mathbb{E}\xi^{r+1} < \infty$, *then* $\mathbb{E}\nu[K]^r < \infty$.

PROOF. We have that $\mathbb{E}\nu[K]^r < \infty$ if $\mathbb{E}\nu^r < \infty$ and $\mathbb{E}(\mu^r | \mu < \infty) < \infty$. The latter holds if $\mathbb{E}\xi^{r+1} < \infty$, and this is a simple consequence of Lemma 5. On the other hand, $\mathbb{E}\nu^r < \infty$ holds if $\mathbb{E}\xi^r < \infty$, as proved in Lemma 6. $\square$

Whereas (C1) and (C2) imply $\mathbb{P}(\nu[K] < \infty) = 1$, we need finite variance for ξ in order that we have finite expectation for $\nu[K]$.

We next construct a further sequence of stopping times,

$$\sigma[1] < \sigma[2] < \cdots,$$

as follows. Assume that (C1) and (C2) hold. Recall that the random variable $\nu[K]$ is a.s. finite; it maps Ω into $\mathbb{N}$. Hence, we can define $\nu[K] \circ \theta^n$ for any $n \in \mathbb{Z}$ and also $\nu[K] \circ \theta^J$ for any measurable $J : \Omega \to \mathbb{Z}$. We define $\sigma[j]$, $j \geq 1$, recursively:

$$\begin{aligned} \sigma[1] &= \nu[K], \\ \sigma[j+1] &= \sigma[j] + \nu[K] \circ \theta^{-\sigma[j]}, \qquad j \geq 1. \end{aligned} \tag{26}$$

Intuitively, given ω, we first construct $\nu[K]$ by (21)–(22) and place a point $\sigma[1]$ at $\nu[K]$. We then shift the origin to $-\nu[K]$ and repeat the recursion with $\omega' = \theta^{-\nu[K]}(\omega)$ in place[2] of ω, thus obtaining a new random variable, $\nu[K] \circ \theta^{-\nu[K]}$. We place another point $\sigma[2]$ at distance[3] $\nu[K] \circ \theta^{-\nu[K]}$ from $\sigma[1]$. The procedure

[2] $\omega' = \theta^{-\nu[K(\omega)](\omega)}(\omega)$.

[3] $\nu[K] \circ \theta^{-\nu[K]}(\omega) = \nu[K(\omega')](\omega') = \nu[K(\theta^{-\nu[K(\omega)](\omega)}(\omega))](\theta^{-\nu[K(\omega)](\omega)}(\omega))$.

continues in the same way. We refer to $-\sigma[1], -\sigma[2], \ldots$ as the sequence of silver points.

LEMMA 11. *Assume that* (C1) *and* (C2) *hold. Define the point process with points* $\sigma[j]$, $j \geq 1$, *as in* (26). *This is a renewal process on* $\mathbb{N}$, *that is, the random variables* $\sigma[1], \sigma[2] - \sigma[1], \sigma[3] - \sigma[2], \ldots$ *are i.i.d. with common distribution* (24).

We are now ready to construct the first gold point Γ_0.

THEOREM 1. *Assume that* (C1) *and* (C2) *hold. Define the sequence* $(\nu[k], \mu[k], k \geq 1)$ *through* (21)–(22) *which is used to define the random variable* $\nu[K]$. *Based on this, define the sequence* $(\sigma[j], j \geq 1)$, *through* (26). *In addition, let*

$$M := \sup_{i \geq 1}\{\xi(i) - i\},$$
$$J := \inf\{j \geq 1 : \sigma[j] \geq M\}.$$

Then

$$\Gamma_0 = -\sigma[J].$$

Before proving the theorem, let us observe that the random variables defined in the theorem statement are a.s.-finite. By (C2), that is, that $\mathbb{E}\xi < \infty$, implies $M < \infty$, a.s.,

$$\mathbb{P}(M \geq m) = \mathbb{P}\big(\xi(i) - i \geq m, \text{ for some } i \geq 1\big)$$

$$\leq \sum_{i=1}^{\infty} \mathbb{P}\big(\xi(i) \geq i + m\big) \tag{27}$$

$$\leq \sum_{i=m+1}^{\infty} \mathbb{P}\big(\xi(i) \geq i\big) \leq \mathbb{E}(\xi, \xi > m) \to 0 \qquad \text{as } m \to \infty. \tag{28}$$

By standard renewal theory, it is easy to see that J, the first exceedance of M by the random walk $(\sigma[j], j \geq 1)$, is also a.s.-finite and, hence, $\sigma[J]$ is an a.s.-finite random variable.

PROOF OF THEOREM 1. Owing to Lemma 9, we have that

$$\text{for all } j \in \mathbb{N}, \qquad \omega \in A^{-}_{-\sigma[j]} \cap A^{+}_{-\sigma[j],0}, \qquad \mathbb{P}\text{-}a.e.\ \omega \in \Omega. \tag{29}$$

Also,

$$\{M \leq \sigma[J]\} = \{\xi(1) \leq \sigma[J] + 1, \xi(2) \leq \sigma[J] + 2, \ldots\}. \tag{30}$$

Fix $n \in \mathbb{N}$ and observe that, from the definition of M and the expressions (10), (12) for $A^+_{-n,0}$ and A^+_{-n}, respectively,

$$\begin{aligned}
&A^-_{-n} \cap A^+_{-n,0} \cap \{M \le n\} \\
&\quad = A^-_{-n} \cap A^+_{-n,0} \cap \{\xi(1) \le n+1, \xi(2) \le n+2, \dots\} \\
&\quad = A^-_{-n} \cap \{\xi(-n+1) \le 1, \dots, \xi(0) \le n, \xi(1) \le n+1, \xi(2) \le n+2, \dots\} \\
&\quad = A^-_{-n} \cap A^+_{-n} \\
&\quad = \{n \in \mathscr{S}\}.
\end{aligned}$$

Combining this with (29) and (30), we obtain

$$-\sigma[J] \in \mathscr{S} \qquad \text{a.s.}$$

It is clear, from the algorithmic construction (21)–(22) of the sequence $(\nu[k], \mu[k], k \ge 1)$, from the algorithmic construction (26) of the $(\sigma[j], j \ge 1)$, and the definition of J, that there can be no point of $\mathscr{S}$ between $-\sigma[J]$ and 0. Therefore, $-\sigma[J]$ is the largest negative point of $\mathscr{S}$. □

REMARK 1. Possible extensions: The algorithmic construction proposed above may be used in a general stationary ergodic framework. In particular, one can easily generalize first-order results (the functional strong law of large numbers). Under reasonable assumptions, one can again prove the finiteness of $\xi(0)$. This will imply the finiteness of $\eta(0)$ and, in turn, the existence of the stationary skeleton. Then the functional strong law of large numbers will follow using well-known tools.

REMARK 2. Simulation and perfect (exact) simulation of the value of the limit C: This depends in a complex way on an infinite number of variables, and one cannot expect an analytic closed form expression. But one can estimate it by running an MCMC algorithm. One can also use the regenerative structure of the model to run the simulation in backward time using the idea of "cycle-truncation" that leads to a simple implementation scheme; cf. [20] for more details. However, each such algorithm gives a biased estimator of the unknown parameter, in general.

In [18], we considered the homogeneous case ($p_j = p$, for all j). In particular, in [18], Section 10 (see also [18], Section 4, for theoretical background), we obtained a stronger result by proposing an algorithm for the perfect simulation of a random sample from an unknown distribution whose mean is the limit C under consideration. The standard MCMC scheme provides an unbiased estimator for this limit.

The ideas behind that algorithm may be efficiently implemented in a number of similar models, for example, in models with long memory (see, e.g., [15]). In fact, in [18], we developed the algorithm for a more general model (we called it "infinite-bin model") and under general stochastic ergodic assumptions.

8. Central limit theorem for the maximum length. Assume now that (C1) holds and

$$\text{(C3)} \qquad \sum_{k=1}^{\infty} k(1-p_1)\cdots(1-p_k) < \infty.$$

From (15) we see that this is equivalent to

$$\text{(C3}'\text{)} \qquad \mathbb{E}\xi^2 < \infty.$$

LEMMA 12. *If* (C1) *and* (C3) *hold*, *then* $\mathbb{E}|\Gamma_0| < \infty$.

PROOF. By Theorem 1, $|\Gamma_0| = \sigma[J] = \min\{\sigma[j] : j \geq 1, \sigma[j] \geq M\}$. Recall that $\sigma[1] < \sigma[2] < \cdots$ are points of a renewal process. This renewal process is clearly independent of $M = \sup_{i \geq 1}\{\xi(i) - i\}$. By standard renewal theory, $\mathbb{E}\sigma[J] < \infty$ if $\mathbb{E}M < \infty$. But the tail of M was estimated in (28). The same inequalities now show that $\mathbb{E}\xi^2 < \infty$ is sufficient for $\mathbb{E}M < \infty$. $\square$

The maximum length L_n of all paths from some $i \geq 0$ to some $j \leq n$ satisfies the following central limit theorem.

THEOREM 2. *Suppose* (C1) *and* (C3) *hold*. *Let*

$$\sigma^2 := \operatorname{var}\big(L(\Gamma_1, \Gamma_2] - C(\Gamma_2 - \Gamma_1)\big).$$

Define

$$\ell_n(t) := \frac{L_{[nt]} - Cnt}{\lambda^{1/2}\sigma\sqrt{n}}, \qquad t \geq 0, n \in \mathbb{N}.$$

Then the sequence of processes ℓ_n, *in the Skorokhod space* $D[0,\infty)$ *equipped with the topology of uniform convergence on compacta* [7], *converges weakly to a standard Brownian motion*.

PROOF. By Lemma 12 we have $\mathbb{E}|\Gamma_0| < \infty$. Hence, $\mathbb{E}\Gamma_1 < \infty$. But the Γ_n form a stationary renewal process. Therefore, $\mathbb{E}\Gamma_1 < \infty$ implies that the variance of $\Gamma_2 - \Gamma_1$ is finite. Since $L(\Gamma_1, \Gamma_2] \leq \Gamma_2 - \Gamma_1$, we have $\sigma^2 < \infty$. The constant C, defined as the a.s.-limit of L_n/n [see (3)], is also finite and nonzero. Lemma 2 shows that $(L_n, n \geq 0)$ is a (stationary) regenerative process. The result then is then obtained by reducing it to Donsker's theorem. This is standard, but we sketch the reduction here for completeness. Let Φ_n be the cardinality of $\mathscr{S} \cap [0,n]$ (the number of Γ_j in the interval $[0,n]$):

$$\Phi_n := |\mathscr{S} \cap [0,n]| = \sum_{j \in \mathbb{Z}} \mathbf{1}(0 \leq \Gamma_j \leq n).$$

So $\Gamma_{\Phi_n} \le n < \Gamma_{\Phi_n+1}$. Write

$$L_{[nt]} = \{L_{[nt]} - L_{\Gamma_{\Phi_{[nt]}}}\} + L_{\Gamma_{\Phi_{[nt]}}},$$
$$nt = \{nt - \Gamma_{\Phi_{[nt]}}\} + \Gamma_{\Phi_{[nt]}}.$$

The quantities in brackets on both lines are tight and so they are negligible when divided by $\sqrt{n}$. So instead of $\ell_n(t)$, we consider

$$\begin{aligned}\widehat{\ell_n}(t) &:= \frac{L_{\Gamma_{\Phi_{[nt]}}} - C\Gamma_{\Phi_{[nt]}}}{\lambda^{1/2}\sigma\sqrt{n}} \\ &= \frac{L_{\Gamma_1} - C\Gamma_1}{\lambda^{1/2}\sigma\sqrt{n}} + \frac{1}{\lambda^{1/2}\sigma\sqrt{n}} \sum_{i=2}^{\Phi_{[nt]}} \{L(\Gamma_{i-1}, \Gamma_i] - C(\Gamma_i - \Gamma_{i-1})\}.\end{aligned} \tag{31}$$

The last term is the one responsible for the weak limit of $\widehat{\ell_n}$ (and hence of ℓ_n). To save some space, put

$$\chi_i := L(\Gamma_{i-1}, \Gamma_i] - C(\Gamma_i - \Gamma_{i-1}).$$

Donsker's theorem says that

$$\left(\frac{1}{\sigma\sqrt{n}} \sum_{i=2}^{nu} \chi_i, u \ge 0\right) \Rightarrow (B_u, u \ge 0),$$

weakly in $D[0, \infty)$, as $n \to \infty$, where B is a standard Brownian motion. Let

$$\varphi_n(t) := \frac{\Phi_{[nt]}}{n}, \qquad t \ge 0.$$

Since φ_n converges weakly, as $n \to \infty$, to the deterministic function $(\lambda t, t \ge 0)$ and since composition is a continuous operation, the continuous mapping theorem tells us that

$$\left(\frac{1}{\sigma\sqrt{n}} \sum_{i=2}^{n\varphi_n(t)} \chi_i, u \ge 0\right) \Rightarrow (B_{\lambda u}, u \ge 0) \stackrel{d}{=} \lambda^{1/2} B,$$

and this readily implies that the last term in (31) converges weakly to a Brownian motion. $\square$

It is now easy to see how the quantity $T[i, j]$, the maximum length of all paths from i to j, behaves. A sufficient condition for $T[i, j]$ to be positive is that there is a skeleton point between i and j. Therefore, keeping i fixed, the probability that eventually for all j sufficiently large $T[i, j] > 0$ is at least equal to the probability that eventually there is a skeleton point in $[i, j]$, and this is certainly equal to one. So, eventually, any two points are connected, a.s.

Moreover,

$$T[\Gamma_i, \Gamma_j] = L[\Gamma_i, \Gamma_j].$$

Indeed, Γ_i is connected to every larger vertex and any vertex smaller than Γ_j is connected to Γ_j. Thus, if a path from some $u \geq \Gamma_i$ to some $v \leq \Gamma_j$ has length $L[\Gamma_i, \Gamma_j]$, we necessarily have $u = \Gamma_i$ and $v = \Gamma_j$ and this shows the equality of the last display.

If n is large enough so that there is at least one skeleton point in $[0, n]$, we have that $0 \rightsquigarrow n$ and

$$L[\Gamma_1, \Gamma_{\Phi_n}] \leq T[0, n] \leq L[\Gamma_0, \Gamma_{\Phi_n+1}],$$

where Φ_n is the number of skeleton points in $[0, n]$. Therefore, we immediately obtain the following:

THEOREM 3. *If* (C1) *and* (C2) *hold*, *then* $T[0, n]/n \to C$, *as* $n \to \infty$, *a.s.*

The same rationale shows the following:

THEOREM 4. *Suppose* (C1) *and* (C3) *hold*. *Then Theorem* 2 *holds with* T *in place of* L.

9. Directed slab graph. Recall that we started with vertex set $V = \mathbb{Z}$ and introduced a random partial order $\rightsquigarrow$ by means of a random directed graph:

$$\begin{aligned} &i \rightsquigarrow j \text{ if } i < j \text{ and } \exists i = i_0 < i_1 < \cdots < i_\ell = j \\ &\text{such that } \alpha_{i_0, i_1} = \cdots = \alpha_{i_{\ell-1}, j} = 1. \end{aligned} \tag{32}$$

A natural generalization is to replace the total order $<$ on the vertex set V by a partial order $\prec$ and substitute the $i < j$ requirement in (32) above by the requirement that $i \prec j$. We here provide an example of such a generalization. A major role in our analysis has been played by the assumption that the underlying probability measure is invariant by some shift θ. Our example will also satisfy this assumption.

Let $(I, \preceq)$ be a finite partially ordered set. We assume that I has a minimum and a maximum, denoted by 0 and M, respectively. In other words, for all $i, j, k \in I$:

(a) $0 \preceq i \preceq i \preceq M$,
(b) if $i \preceq j \preceq i$ then $i = j$,
(c) if $i \preceq j \preceq k$ then $i \preceq k$.

Consider $V = \mathbb{Z} \times I$. We call this vertex set a *cylinder*. In the case $I = \{0, 1, \ldots, M\}$, with the usual ordering, we call V a *slab*. Elements of V will be denoted by (x, i), (y, j), etc. We introduce the component-wise partial ordering $\ll$ on V by

$$(x, i) \ll (y, j) \iff (x, i) \neq (y, j) \quad \text{and} \quad x \leq y, i \preceq j,$$

and write $(y, j) \gg (x, i)$ for the same thing. Next, we assign an edge $((x, i), (y, j))$ to each pair of vertices such that $(x, i) \ll (y, j)$ with probability $r_{y-x,i,j}$, independently from pair to pair. This is done by means of random variables $\alpha_{(x,i),(y,j)}$:

$$\mathbb{P}\big(\alpha_{(x,i),(y,j)} = 0\big) = 1 - \mathbb{P}\big(\alpha_{(x,i),(y,j)} = -\infty\big) r_{y-x,i,j}.$$

We shall make this more formal in the sequel. The problem is, again, the behavior of a longest path from (x, i) to (y, j). This length is denoted by $T[(x, i), (y, j)]$. We also define $L[(x, i), (y, j)]$ to be the maximum length of all paths starting from some $(x', i') \gg (x, i)$ and ending at some $(y', j') \ll (y, j)$.

An appropriate probability space for the model is now described. Let $\boldsymbol{\delta} = (\delta_{x,i,j}, x \in \mathbb{Z}, i, j \in I)$ be a collection of independent $\{-\infty, 1\}$-valued random variables with

$$\mathbb{P}(\delta_{x,i,j} = 1) = r_{x,i,j},$$

assuming that $r_{x,i,j} = 0$ if $x \le 0$ or if $i \succ j$. Next, let $\boldsymbol{\delta}^{(x)}$, $x \in \mathbb{Z}$, be a collection of i.i.d. copies of $\boldsymbol{\delta}$. The probability space Ω is defined to contain infinite vectors $\omega = (\boldsymbol{\delta}^{(x)}, x \in \mathbb{Z})$. In other words, let $\Omega = (\{-\infty, 1\}^{\mathbb{Z}\times I\times I})^{\mathbb{Z}}$ with $\{-\infty, 1\}^{\mathbb{Z}\times I\times I}$ be the space of values of each $\boldsymbol{\delta}^{(x)}$, and with $\mathbb{P}$ being a product measure. A shift θ on Ω is taken to be the natural map

$$\omega = \big(x \mapsto \boldsymbol{\delta}^{(x)}\big) \mapsto \theta\omega = \big(x \mapsto \boldsymbol{\delta}^{(x+1)}\big). \tag{33}$$

Clearly, $\mathbb{P}$ is preserved by θ. The random variables $\alpha_{(x,i),(y,j)}$ are now given by

$$\alpha_{(x,i),(y,j)}(\omega) = \delta^{(x)}_{y-x,i,j}$$

and it is easy to check their θ-compatibility: $\alpha_{(x,i),(y,j)}(\theta\omega) = \alpha_{(x+1,i),(y+1,j)}(\omega)$.

We introduce the following assumptions on the probabilities $r_{x,i,j}$:

(D0) $r_{x,i,i} =: p_x$ for all $i \in I$,

(D1) $0 < p_1 < 1$,

(D2) $\displaystyle\sum_{x=1}^{\infty} (1 - p_1) \cdots (1 - p_x) < \infty$,

(D2′) $\displaystyle\sum_{x=1}^{\infty} x(1 - p_1) \cdots (1 - p_x) < \infty$,

(D3) for all $i, j \in I$ with $i \prec j$, we have $r_{0,i,j} > 0$.

Of these, the last one is not an essential condition. It is only introduced for convenience. We will comment on it later. Of course, (D2′) is stronger than (D2) and it will be used for the proof of the CLT.

9.1. *The random graph* $G[x, y]$. The random directed graph $G = (V, E)$ with $V = \mathbb{Z} \times I$ and E consisting of all $((x, i), (y, j))$ such that $\alpha_{(x,i),(y,j)} = 1$ is now a well-defined object. Let $G[x, y]$ be the restriction of G on the vertex set $[x, y] \times I$ where $x \leq y$ are two integers. Let

$$L[x, y] := \max_{\substack{x \leq x' \leq y' \leq y \\ i,j \in I}} L[(x', i), (y', j)]$$

be the maximum length of all paths in $G[x, y]$. We have θ-compatibility

$$L[x, y] \circ \theta = L[x + 1, y + 1],$$

and, by an argument analogous to the one used to obtain (2), we have the subadditivity property

$$L[x, z] \leq L[x, y] + L[y, z] + 1, \qquad x \leq y \leq z.$$

Therefore,

$$L_N/N := L[0, N]/N \to C \qquad \text{as } n \to \infty, \text{ a.s.},$$

for some deterministic constant C which, under the assumption (D1), is positive. One can show that, under assumptions (D0)–(D3), the constant C is identical to the one of (3) for the line graph.

9.2. *The random graph* $G^{(i)}$. Let $G^{(i)}$ be the restriction of G on the vertex set $V \times \{i\}$, $i \in I$. It is clear that each $G^{(i)}$ is a line model as studied earlier. In fact, the $G^{(i)}$, $i \in I$ are i.i.d. We denote by $L^{(i)}[x, y]$ the maximum length of all paths of $G^{(i)}$ from some vertex $x' \geq x$ to some vertex $y' \leq y$. We shall let $\mathscr{S}^{(i)}$ be the skeleton of $G^{(i)}$. Then, assuming (D1) and (D2), each $\mathscr{S}^{(i)}$ forms a stationary renewal process with a nontrivial rate. Moreover, (D1) implies that this renewal process is aperiodic.

10. Central limit theorem for the directed cylinder graph. We first describe the limiting process. To do this, we need the following. First, let $(B^{(i)}(t), t \geq 0)$, $i \in I$, be i.i.d. standard Brownian motions, all starting from 0. Second, let $H(I, \preceq)$ be the *Hasse diagram* [16] corresponding to the partially ordered set I. This is a directed graph with vertex set I and an edge from i to j, with $i \preceq j$, if there is no k, distinct from i and j, such that $i \preceq k \preceq j$. Let $\iota = (\iota_0, \iota_1, \dots, \iota_r)$ be a path in $H(I, \preceq)$ starting from $\iota_0 = 0$ and ending at $\iota_r = M$. The length of the path is $r = |\iota|$. For each such path ι, define the stochastic process $(Z^{(\iota)}(t), t \geq 0)$ by

$$Z^{(\iota)}(t) := \sup\bigl\{B^{(\iota_0)}(t_0) + \bigl[B^{(\iota_1)}(t_1) - B^{(\iota_1)}(t_0)\bigr] + \cdots + \bigl[B^{(\iota_{|\iota|})}(t_{|\iota|}) - B^{(\iota_{|\iota|})}(t_{|\iota|-1})\bigr]\bigr\}, \tag{34}$$

where the supremum is taken over all $0 \leq t_0 \leq t_1 \leq \cdots \leq t_{|\iota|} = t$. Then let

$$Z(t) := \max_{\iota} Z^{(\iota)}(t), \tag{35}$$

where the maximum is taken over all paths ι from the minimum to the maximum element in the Hasse diagram $H(I, \preceq)$.

The main theorem of this section is as follows:

THEOREM 5. *Let G be a directed cylinder graph and assume that* (D0), (D1), (D2′), (D3) *hold. Let L_n be the maximum length of all paths in $G[0, n]$. There exists a constant $\kappa > 0$ such that*

$$\ell_n(t) := \frac{L_{[nt]} - Cnt}{\kappa\sqrt{n}}, \qquad t \ge 0, n \in \mathbb{N},$$

converges weakly, as $n \to \infty$, in the Skorokhod space $D[0, \infty)$ equipped with the topology of uniform convergence on compacta, to the stochastic process Z defined in (34)–(35).

PROOF. Since the $\mathscr{S}^{(i)}$, $i \in I$, are independent aperiodic renewal processes, we have that

$$\mathscr{S} := \left\{x \in \mathbb{Z} : x \in \bigcap_{i \in I} \mathscr{S}^{(i)}, \alpha_{(x,i),(x,j)} = 1 \text{ for all } i, j \in I \text{ with } i \prec j\right\}$$

is also a renewal process. Indeed, Lindvall [27] shows that $\bigcap_i \mathscr{S}^{(i)}$ is a stationary renewal process. Now $\mathscr{S}$ is obtained from $\bigcap_{i \in I} \mathscr{S}^{(i)}$ by a further independent thinning with positive probability due to the convenient assumption (D3). Condition (D2) implies that the rate of each $\mathscr{S}^{(i)}$ is positive and this implies that the rate of $\bigcap_{i \in I} \mathscr{S}^{(i)}$ is positive. Hence, the rate of $\mathscr{S}$ is also positive. Call this rate λ. We have $0 < \lambda \le 1$. Moreover, $\mathscr{S}$ is stationary: $\mathscr{S} \circ \theta = \mathscr{S}$. Enumerate now the points of $\mathscr{S}$ by

$$\cdots < \Gamma_{-1} < \Gamma_0 < 0 \le \Gamma_1 < \Gamma_2 < \cdots.$$

We have $\mathbb{E}(\Gamma_2 - \Gamma_1) = 1/\lambda$. If

$$\Phi_n := |\mathscr{S} \cap [0, n]|,$$

we have $\lim_{n\to\infty} \Phi_n/n = \lambda$, a.s. Furthermore, $C = \lambda \mathbb{E} L[\Gamma_1, \Gamma_2] \le 1$. Condition (D2′) implies that $\mathbb{E}(\Gamma_2 - \Gamma_1)^2 < \infty$ and, hence, $\mathbb{E} L^{(i)}[\Gamma_2 - \Gamma_1]^2 < \infty$. By Corollary 1, the random variables

$$(\Gamma_2 - \Gamma_1, L^{(i)}[\Gamma_1, \Gamma_2]), \qquad (\Gamma_3 - \Gamma_2, L^{(i)}[\Gamma_3, \Gamma_2]), \ldots$$

are i.i.d., and since $\mathscr{S}$ is obtained by independent thinning of $\bigcap_{i \in I} \mathscr{S}^{(i)}$, we further have that the rows of the last display are also independent when i ranges in I.

Consider next a path $\iota = (\iota_0, \iota_1, \ldots, \iota_r)$, of length $|\iota| = r$ in the Hasse diagram $H(I, \preceq)$, and define the quantities

$$\begin{aligned} L^*(\iota)_n &:= \max_{1 \le j_0 \le j_1 \le \cdots \le j_r = n} \{L^{(\iota_0)}[\Gamma_1, \Gamma_{j_0}] + L^{(\iota_1)}[\Gamma_{j_0}, \Gamma_{j_1}] + \cdots \\ &\qquad\qquad + L^{(\iota_r)}[\Gamma_{j_{r-1}}, \Gamma_{j_r}]\}, \\ L_n^* &:= \max_\iota L^*(\iota)_n, \end{aligned}$$

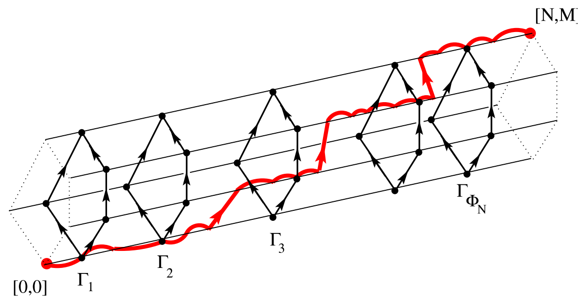

FIG. 2. *The skeleton for the slab graph and a longest path.*

where the last maximum is taken over all paths ι from the minimum to the maximum element of the Hasse diagram.

We now argue that the quantity of interest L_n is of order $L_n^* + o_{\mathrm{d}}(1)$ when n is large by providing an upper and a lower bound. The key observation is that when n is large, the number of points $\Gamma_j \le n$ grows at a positive rate (and hence to infinity). At each of these points, say, Γ_j, the graph $G[\Gamma_j, \Gamma_j]$ (being a vertical slice of G—see Figure 2) is precisely the Hasse diagram:

$$G[\Gamma_j, \Gamma_j] = H(I, \preceq), \qquad j \in \mathbb{Z}.$$

Fix $\iota' \prec \iota''$ in I. Since Γ_j is a point in the skeleton of $G^{(\iota')}$, any $x \le \Gamma_j$ is connected to Γ_j in $G^{(\iota')}$. Similarly, Γ_j is connected to any y in $G^{(\iota'')}$. Since ι' is connected to ι'' in $G[\Gamma_j, \Gamma_j]$, it follows that, almost surely, there is a path in G from any (x, ι') to any (y, ι''), if $x \le \Gamma_j \le y$ for some $\Gamma_j \in \mathscr{S}$ and if $\iota' \prec \iota''$.

Assume that $\Phi_n \ge 2$. Let $\iota = (\iota_0, \iota_1, \dots, \iota_r)$ be a path in $H(I, \preceq)$ with $\iota_0 = 0$, $\iota_r = M$ and consider integers

$$1 \le j_0 \le j_1 \le \cdots \le j_{r-1} \le j_r = \Phi_n. \tag{36}$$

Keep in mind that

$$\Gamma_{\Phi_n} \le n.$$

By the construction of the set $\mathscr{S}$, the following is true:

$$\begin{aligned}(\Gamma_1, 0) &= (\Gamma_1, \iota_0) \rightsquigarrow (\Gamma_{j_0}, \iota_0) \rightsquigarrow (\Gamma_{j_0}, \iota_1) \rightsquigarrow (\Gamma_{j_1}, \iota_1) \rightsquigarrow \cdots \\ &\rightsquigarrow (\Gamma_{j_{r-1}}, \iota_r) \rightsquigarrow (\Gamma_{j_r}, \iota_r) = (\Gamma_{\Phi_n}, M),\end{aligned}$$

where $(x, \iota') \rightsquigarrow (y, \iota'')$ means that there is a path from (x, ι') to (y, ι'') in G. Therefore,

$$L_n \ge L^{(\iota_0)}[\Gamma_1, \Gamma_{j_0}] + L^{(\iota_1)}[\Gamma_{j_0}, \Gamma_{j_1}] + \cdots + L^{(\iota_r)}[\Gamma_{j_{r-1}}, \Gamma_{j_r}],$$

because the right-hand side is a lower bound on the length of the specific path chosen in the last display. By keeping ι fixed and maximizing over the $j_0, \dots, j_r$

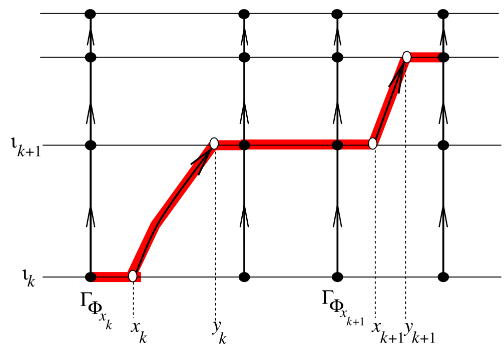

FIG. 3. *Construction used in obtaining the upper bound.*

satisfying (36), we obtain $L_n \geq L^*(\iota)_n$, and by maximizing over ι, we obtain the lower bound

$$L_n \geq L^*_{\Phi_n}.$$

To obtain an upper bound, let π^* be a path that achieves the maximum in L_n. Assume that $\Phi_n \geq 1$ so that, by the key observation above, $(0,0)$ is connected to (n, M) in G. See Figure 3. Hence, π^* is necessarily a path from $(0,0)$ to (n, M).

Let

$$0 = \iota_0 \prec \iota_1 \prec \cdots \prec \iota_s = M$$

be the distinct values of the I-components of the elements of π^* in order of appearance in π^*. [The sequence $(\iota_0, \iota_1, \dots, \iota_s)$ is not necessarily a path in $H(I, \preceq)$.] So for each $k = 0, \dots, s-1$, there are vertices (x_k, ι_k), (y_k, ι_{k+1}) which are consecutive in the path π^*. Hence,

$$x_k \leq y_k \leq x_{k+1} \qquad \text{for all } k = 0, 1, \dots, s-1,$$

where, by convention, we set $x_s = n$. The point of $\mathscr{S}$ prior to x_k is $\Gamma_{\Phi_{x_k}}$ and, since π^* has maximum length, $(\Gamma_{\Phi_{x_k}}, \iota_k)$ is an element of π^*. By the maximality of π^* again, we have that x_k and y_k are contained between two successive points of $\mathscr{S}$ (otherwise we would be able to strictly increase the length of the path). Hence,

$$\Gamma_{\Phi_{x_k}} \leq x_k \leq y_k \leq \Gamma_{1+\Phi_{x_k}} \leq x_{k+1} \qquad \text{for all } k = 0, 1, \dots, s-1. \tag{37}$$

We thus have

$$\begin{aligned} L_n = |\pi^*| &= L^{(\iota_0)}[0, \Gamma_1] + L^{(\iota_0)}[\Gamma_1, \Gamma_{\Phi_{x_0}}] \\ &\quad + \sum_{k=0}^{s-1} \{ L^{(\iota_k)}[\Gamma_{\Phi_{x_k}}, x_k] + 1 + L^{(\iota_{k+1})}[y_k, \Gamma_{\Phi_{x_{k+1}}}] \} + L^{(\iota_s)}[\Gamma_{\Phi_n}, n]. \end{aligned}$$

Due to (37), we have

$$L^{(\iota_k)}[\Gamma_{\Phi_{x_k}}, x_k] \le L^{(\iota_k)}[\Gamma_{\Phi_{x_k}}, \Gamma_{1+\Phi_{x_k}}], \tag{38}$$

$$L^{(\iota_{k+1})}[y_k, \Gamma_{\Phi_{x_{k+1}}}] \le L^{(\iota_{k+1})}[\Gamma_{\Phi_{x_k}}, \Gamma_{\Phi_{x_{k+1}}}], \qquad k = 0, \dots, s-1. \tag{39}$$

Moreover,

$$L^{(\iota_0)}[0, \Gamma_1] \le L^{(\iota_0)}[\Gamma_0, \Gamma_1], \tag{40}$$

$$L^{(\iota_s)}[\Gamma_{\Phi_n}, n] \le L^{(\iota_s)}[\Gamma_{\Phi_n}, \Gamma_{1+\Phi_n}]. \tag{41}$$

Each of the right-hand sides of (38), (40) and (41) is bounded above by $\max_{0 \le j \le \Phi_n} L^{(\iota)}[\Gamma_j, \Gamma_{1+j}]$. If we then define

$$\zeta_n := \sum_{\iota \in I} \max_{0 \le j \le \Phi_n} L^{(\iota)}[\Gamma_j, \Gamma_{1+j}]$$

and use (39), we obtain

$$L_n \le \zeta_n + M + \sum_{k=0}^{s-1} L^{(\iota_{k+1})}[\Gamma_{\Phi_{x_k}}, \Gamma_{\Phi_{x_{k+1}}}].$$

Since for each sequence $0 = \iota_0 \prec \iota_1 \prec \cdots \prec \iota_s = M$ of distinct ordered elements of I we can find a path in the Hasse diagram containing these elements, it follows easily that

$$L_n \le \zeta_n + M + L^*_{\Phi_n},$$

which gives the upper bound. The upper bound is close to L_n in the sense that the sequence ζ_n are of order 1 in distribution, that is, that (ζ_n) is a tight random sequence. On the other hand, $nt = \Gamma_{\Phi_{[nt]}} - \Gamma_1 + o_{\mathrm{d}}(1)$. It is thus clear that the weak limit of ℓ_n and that of

$$\ell^*_n(t) := \frac{L^*_{\Phi_{[nt]}} - C(\Gamma_{\Phi_{[nt]}} - \Gamma_1)}{\kappa\sqrt{n}}, \qquad t \ge 0,$$

if it exists, will be identical. Setting

$$\ell^{**}_n(t) := \frac{L^*_{[nt]} - C(\Gamma_{[nt]} - \Gamma_1)}{\kappa\sqrt{n}}, \qquad \varphi_n(t) := \frac{\Phi_{[nt]}}{n},$$

we have

$$\ell^*_n(t) = \ell^{**}_n(\varphi_n(t)), \tag{42}$$

and so the weak limit of ℓ^*_n is equal to that of ℓ^{**}_n (if this exists) composed by the function $\{\lambda t\}$.

To show that the weak limit of ℓ_n^* exists and to find it, define the function $\psi : D[0,\infty)^I \to D[0,\infty)$ by

$$\psi\big(\beta^{(i)}, i \in I\big)(t) := \max_{\iota} \sup_{\substack{0 \le t_0 \le t_1 \le \cdots \le t_r = t \\ |\iota| = r}} \big\{\beta^{(\iota_0)}(t_0) + \big[\beta^{(\iota_1)}(t_1) - \beta^{(\iota_1)}(t_0)\big] + \cdots \\ + \big[\beta^{(\iota_r)}(t_r) - \beta^{(\iota_r)}(t_{r-1})\big]\big\},$$

where the maximum is taken over all paths ι from the minimum to the maximum element in the Hasse diagram $H(I, \preceq)$. The function ψ is continuous (with respect to the topology of uniform convergence). Let

$$s_n^{(i)}(t) := \frac{L^{(i)}[\Gamma_1, \Gamma_{[nt]}] - C(\Gamma_{[nt]} - \Gamma_1)}{\sigma\sqrt{n}}, \qquad t \ge 0, i \in I,$$

where

$$\sigma^2 := \mathrm{var}\big\{L^{(i)}[\Gamma_1, \Gamma_2] - C(\Gamma_2 - \Gamma_1)\big\}.$$

Since $L^{(i)}[\Gamma_j, \Gamma_{j+1}]$, $j \ge 1$, $i \in I$, are i.i.d. with common variance σ^2, we have (Theorem 2) that

$$\big(s_n^{(i)}, i \in I\big) \Rightarrow \big(B^{(i)}, i \in I\big), \tag{43}$$

where $B^{(i)}$, $i \in I$, are i.i.d. standard Brownian motions. Let

$$\kappa := \lambda^{1/2}\sigma$$

and observe that

$$\ell_n^{**}(t) = \lambda^{-1/2} \cdot \psi\big(s_n^{(i)}, i \in I\big)(t).$$

By (43) and the invariance principle,

$$\ell_n^{**} \Rightarrow \lambda^{-1/2} \cdot \psi\big(B^{(i)}, i \in I\big).$$

By the relation (42) and the remark following it, we have

$$\ell_n^* \Rightarrow \psi\big(B^{(i)}, i \in I\big),$$

and the right-hand side is equal in distribution to Z [defined by (34)–(35)]. $\square$

The remarks at the end of Section 8 also apply in the current case. We can easily conclude that T_n, the maximum length of all paths from $(0,0)$ to (n, M), has the same asymptotics as L_n. In particular, Theorem 5 holds if we replace L_n by T_n.

11. Connection to last passage percolation. Consider now the case

$$I = \{0, 1, \dots, M\}$$

with the usual ordering. Assumption (D3) can be substituted by

$$\text{(D3}'\text{)} \quad \text{for all } 1 \le i \le M \text{ we have } r_{0,i-1,i} > 0.$$

Let G_M be the corresponding random directed cylinder graph, referred to as slab graph here. In particular, we can think of G_M as the restriction of a graph G_∞ on the vertex set $\mathbb{Z} \times \mathbb{Z}_+$, where two vertices (x, i) and (y, j), with $(x, i) \ll (y, j)$, are connected with probability $p_{y-x,j-i}$ that depends on the relative position of the two vertices on the 2-dimensional lattice.

The problem here becomes that of a last passage percolation, although the model is not the standard nearest-neighbor one. Physically, we can think of tunnels which run upward (or in directions southwest to northeast) and fluid moving in tunnels. It takes one unit of time to cross a specific tunnel. We are interested in the particle that starts from $(0, 0)$ and reaches (n, M) in the largest possible time. Since the Hasse diagram of the set $\{0, 1, \dots, M\}$ with the natural ordering is the linear graph with edges from $i - 1$ to i, $1 \le i \le M$, the limit process Z is given by the simplified expression

$$Z(t) = \max_{0 \le t_0 \le \cdots \le t_M = t} \bigl\{B^{(0)}(t_0) + \bigl[B^{(1)}(t_1) - B^{(1)}(t_0)\bigr] + \cdots \\ + \bigl[B^{(M)}(t_M) - B^{(M)}(t_{M-1})\bigr]\bigr\}, \qquad t \ge 0.$$

The latter process is a Brownian last passage percolation process. As was shown in [6, 22, 33], it is a non-Gaussian process with marginal distribution

$$Z(t) \stackrel{d}{=} \sqrt{t} \cdot \lambda_M,$$

for each $t \ge 0$, where λ_M is the largest eigenvalue of a random $(M + 1) \times (M + 1)$ matrix from the Gaussian Unitary Ensemble (GUE) [30].

Tracy and Widom [35, 36] showed that, as $M \to \infty$, the following weak limit holds:

$$M^{1/6}\bigl(\lambda_M - 2\sqrt{M}\bigr) \Rightarrow F_{\mathrm{TW}},$$

with F_{TW} being the Tracy–Widom distribution whose hazard rate equals $\int_t^\infty q(x)^2\,dx$, where $q(x)$ satisfies a Painlevé II equation; see [2], equation (3.1.7). For an account on the universality of this distribution, see, for example, [17]. A number of interesting results have been proved relating this limiting distribution with certain stochastic models. These models include longest increasing subsequence [4], last passage percolation, noncolliding particles, tandem queues [6, 22] and random tilings [25]. For the last passage percolation, in particular, this limit is known to appear in two cases. The first is the Brownian last passage percolation. The second is the last passage percolation model with exponential (or geometric) weights. In this model one puts independent and identically distributed exponential random variables in the vertices of $\mathbb{Z}_+^2$ and considers the maximum $L(M, N)$ of the sums of the weights over all directed paths from $(0, 0)$ to (M, N). It was shown in [24] that the random variable $L(N, N)$, properly normalized, converges to the Tracy–Widom distribution as N goes to infinity. In [8], more general weights were considered and an analogous result for the random variable $L(N, N^a)$ (for an appropriate a depending on moment conditions) was obtained. It is then natural to conjecture that a similar phenomenon occurs in our slab graph too. We also note that results similar to that of Theorem 5 hold in various classes of stochastic networks and, in particular, for the stationary sojourn time in tandem queues (see, e.g., [21]).

Acknowledgments. We thank the Isaac Newton Institute for Mathematical Sciences for providing the stimulating research atmosphere where this research work was completed. We are also grateful to Svante Janson and Graham Brightwell for pointing out reference [1] to us.

REFERENCES

[1] ALON, N., BOLLOBÁS, B., BRIGHTWELL, G. and JANSON, S. (1994). Linear extensions of a random partial order. *Ann. Appl. Probab.* **4** 108–123. MR1258175

[2] ANDERSON, G. W., GUIONNET, A. and ZEITOUNI, O. (2010). *An Introduction to Random Matrices*. *Cambridge Studies in Advanced Mathematics* **118**. Cambridge Univ. Press, Cambridge. MR2760897

[3] BAIK, J. (2005). Limiting distribution of last passage percolation models. In *XIVth International Congress on Mathematical Physics* 339–346. World Sci. Publ., Hackensack, NJ. MR2227847

[4] BAIK, J., DEIFT, P. and JOHANSSON, K. (1999). On the distribution of the length of the longest increasing subsequence of random permutations. *J. Amer. Math. Soc.* **12** 1119–1178. MR1682248

[5] BARAK, A. B. and ERDŐS, P. (1984). On the maximal number of strongly independent vertices in a random acyclic directed graph. *SIAM J. Algebraic Discrete Methods* **5** 508–514. MR0763980

[6] BARYSHNIKOV, Y. (2001). GUEs and queues. *Probab. Theory Related Fields* **119** 256–274. MR1818248

[7] BILLINGSLEY, P. (1968). *Convergence of Probability Measures*. Wiley, New York. MR0233396

[8] BODINEAU, T. and MARTIN, J. (2005). A universality property for last-passage percolation paths close to the axis. *Electron. Commun. Probab.* **10** 105–112 (electronic). MR2150699

[9] BOROVKOV, A. A. (1978). Ergodicity and stability theorems for a class of stochastic equations and their applications. *Theory Probab. Appl.* **23** 227–258.

[10] BOROVKOV, A. A. (1980). *Asymptotic Methods in Queueing Theory*, Russian ed. Nauka, Moscow. Revised English edition: Wiley, New York, 1984.

[11] BOROVKOV, A. A. (1998). *Ergodicity and Stability of Stochastic Processes*. Wiley, Chichester. MR1658404

[12] BOROVKOV, A. A. and FOSS, S. G. (1992). Stochastically recursive sequences and their generalizations. *Siberian Adv. Math.* **2** 16–81. MR1157423

[13] BOROVKOV, A. A. and FOSS, S. G. (1994). Two ergodicity criteria for stochastically recursive sequences. *Acta Appl. Math.* **34** 125–134. MR1273850

[14] COHEN, J. E. and NEWMAN, C. M. (1991). Community area and food chain length: Theoretical predictions. *Amer. Naturalist* **138** 1542–1554.

[15] COMETS, F., FERNÁNDEZ, R. and FERRARI, P. A. (2002). Processes with long memory: Regenerative construction and perfect simulation. *Ann. Appl. Probab.* **12** 921–943. MR1925446

[16] DAVEY, B. A. and PRIESTLEY, H. A. (1990). *Introduction to Lattices and Order*. Cambridge Univ. Press, Cambridge. MR1058437

[17] DEIFT, P. (2007). Universality for mathematical and physical systems. In *International Congress of Mathematicians. Vol. I* 125–152. Eur. Math. Soc., Zürich. MR2334189

[18] FOSS, S. and KONSTANTOPOULOS, T. (2003). Extended renovation theory and limit theorems for stochastic ordered graphs. *Markov Process. Related Fields* **9** 413–468. MR2028221

[19] FOSS, S. G. and TWEEDIE, R. L. (1998). Perfect simulation and backward coupling. *Comm. Statist. Stochastic Models* **14** 187–203. MR1617572

[20] FOSS, S. G., TWEEDIE, R. L. and CORCORAN, J. N. (1998). Simulating the invariant measures of Markov chains using backward coupling at regeneration times. *Probab. Engrg. Inform. Sci.* **12** 303–320. MR1631374

[21] GLYNN, P. W. and WHITT, W. (1991). Departures from many queues in series. *Ann. Appl. Probab.* **1** 546–572. MR1129774

[22] GRAVNER, J., TRACY, C. A. and WIDOM, H. (2001). Limit theorems for height fluctuations in a class of discrete space and time growth models. *J. Stat. Phys.* **102** 1085–1132. MR1830441

[23] ISOPI, M. and NEWMAN, C. M. (1994). Speed of parallel processing for random task graphs. *Comm. Pure Appl. Math.* **47** 361–376. MR1266246

[24] JOHANSSON, K. (2000). Shape fluctuations and random matrices. *Comm. Math. Phys.* **209** 437–476. MR1737991

[25] JOHANSSON, K. (2005). The arctic circle boundary and the Airy process. *Ann. Probab.* **33** 1–30. MR2118857

[26] KALLENBERG, O. (1992). *Foundations of Modern Probability*. Springer, New York.

[27] LINDVALL, T. (1992). *Lectures on the Coupling Method*. Wiley, New York. MR1180522

[28] MARTIN, J. B. (2004). Limiting shape for directed percolation models. *Ann. Probab.* **32** 2908–2937. MR2094434

[29] MARTIN, J. B. (2006). Last-passage percolation with general weight distribution. *Markov Process. Related Fields* **12** 273–299. MR2249632

[30] MEHTA, M. L. (2004). *Random Matrices*, 3rd ed. *Pure and Applied Mathematics* (*Amsterdam*) **142**. Elsevier/Academic Press, Amsterdam. MR2129906

[31] NEWMAN, C. M. (1992). Chain lengths in certain random directed graphs. *Random Structures Algorithms* **3** 243–253. MR1164838

[32] NEWMAN, C. N. and COHEN, J. E. (1986). A stochastic theory of community food webs: IV. Theory of food chains in large webs. *Proc. R. Soc. London Ser. B* **228** 355–377.

[33] O'CONNELL, N. and YOR, M. (2002). A representation for noncolliding random walks. *Electron. Commun. Probab.* **7** 1–12 (electronic). MR1887169

[34] PROPP, J. and WILSON, D. (1997). Coupling from the past: A user's guide. In *Microsurveys in Discrete Probability: DIMACS Workshop, June* 2–6, 1997. Amer. Math. Soc., Providence, RI.

[35] TRACY, C. A. and WIDOM, H. (1993). Level-spacing distributions and the Airy kernel. *Phys. Lett. B* **305** 115–118. MR1215903

[36] TRACY, C. A. and WIDOM, H. (1994). Level-spacing distributions and the Airy kernel. *Comm. Math. Phys.* **159** 151–174. MR1257246

D. DENISOV
SCHOOL OF MATHEMATICS
CARDIFF UNIVERSITY
CARDIFF CF24 4AG
UNITED KINGDOM
E-MAIL: dennissov@googlemail.com

S. FOSS
SCHOOL OF MATHEMATICAL AND COMPUTER SCIENCES
HERIOT–WATT UNIVERSITY
EDINBURGH EH14 4AS
UNITED KINGDOM
E-MAIL: foss@ma.hw.ac.uk

T. KONSTANTOPOULOS
DEPARTMENT OF MATHEMATICS
UPPSALA UNIVERSITY
P.O. BOX 480
751 06 UPPSALA
SWEDEN
E-MAIL: takis@math.uu.se
URL: http://www.math.uu.se/~takis