J Theor Probab
DOI 10.1007/s10959-012-0434-3

Iterating Brownian Motions, Ad Libitum

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Received: 6 January 2012 / Revised: 19 June 2012

Abstract Let $B_1, B_2, \ldots$ be independent one-dimensional Brownian motions parameterized by the whole real line such that $B_i(0) = 0$ for every $i \geq 1$. We consider the nth iterated Brownian motion $W_n(t) = B_n(B_{n-1}(\cdots(B_2(B_1(t)))\cdots))$. Although the sequence of processes $(W_n)_{n\geq 1}$ does not converge in a functional sense, we prove that the finite-dimensional marginals converge. As a consequence, we deduce that the random occupation measures of W_n converge to a random probability measure μ_∞. We then prove that μ_∞ almost surely has a continuous density which should be thought of as the local time process of the infinite iteration W_∞ of independent Brownian motions. We also prove that the collection of random variables $(W_\infty(t), t \in \mathbb{R} \setminus \{0\})$ is exchangeable with directing measure μ_∞.

Keywords Brownian motion · Iterated Brownian motion · Harris chain · Random measure · Exchangeability · Weak convergence · Local time · de Finetti–Hewitt–Savage theorem

Mathematics Subject Classification (2000) Primary 60J65 · 60J05 · Secondary 60G57 · 60E99

1 Introduction

Let $B_+ = (B_+(t), t \geq 0)$ and $B_- = (B_-(t), t \geq 0)$ be independent standard one-dimensional Brownian motions starting from 0. The process $B(t) := B_+(t)$ if $t \geq 0$

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Published online: 19 July 2012

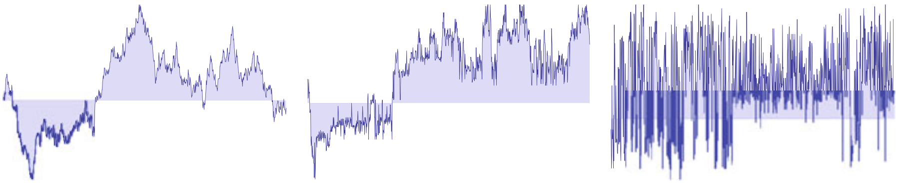

Fig. 1 First, second, and third iterations of Brownian motions

and $B(t) := B_-(-t)$ if $t \le 0$ is called a two-sided Brownian motion. In this paper we study the iterations of independent (two-sided) Brownian motions. Formally, let $B_1, B_2, \dots$ be a sequence of independent two-sided Brownian motions and, for every $n \ge 1$ and every $t \in \mathbb{R}$, set

$$W_n(t) := B_n\bigl(B_{n-1}\bigl(\cdots\bigl(B_2\bigl(B_1(t)\bigr)\bigr)\cdots\bigr)\bigr).$$

Burdzy [5] studied sample path properties of the random function $t \mapsto W_2(t)$ and coined the terminology of (second) "iterated Brownian motion" for this object. A motivation for this study is that iterated Brownian motion can be used to construct a solution to the fourth-order PDE $\partial u/\partial t = \frac{1}{8}\partial^4 u/\partial x^4$; see [7]. This model has triggered a lot of work, see [3–6, 12] and the references therein.

Another motivation is that the process W_n is not a semimartingale (unless $n = 1$). Indeed, for $n = 2$, a simple calculation shows that the quadratic variation of $W_2 = B_2 \circ B_1$ does not exist, but its quartic variation does. Similarly the 2^n-variation of W_n is finite for $n \ge 1$. Hence, as n increases, the process W_n becomes wilder and wilder. Also, W_n is self-similar with index 2^{-n}, i.e.,

$$\bigl(W_n(\alpha t), t \in \mathbb{R}\bigr) \stackrel{(d)}{=} \bigl(\alpha^{2^{-n}} W_n(t), t \in \mathbb{R}\bigr)$$

for all $\alpha > 0$. See Fig. 1 for a comparison of sample paths of W_n, $n = 1, 2, 3$. All these reasons make one suspect that convergence of the laws of the W_ns in a "nice" function space (e.g., the space of continuous functions) is impossible (see Remark 1).

However, following the principle of Berman saying that wild functions must have smooth local times (see the nice survey [8]), we prove that the occupation measures μ_n of W_n over [0, 1] converge in distribution to a random measure μ_∞ which can be interpreted as the occupation measure of the infinite iteration "W_∞". More precisely, let μ_n be the random probability measure defined by

$$\int_{\mathbb{R}} \mathrm{d}\mu_n\, f = \int_0^1 \mathrm{d}t\, f\bigl(W_n(t)\bigr) \tag{1}$$

for every Borel-measurable function $f : \mathbb{R} \to \mathbb{R}_+$. Our main result is a limit theorem for the sequence $(\mu_n)_{n \ge 1}$. Restricting the integration to the unit interval is convenient and poses no loss of generality due to the self-similarity property of W_n.

Let $\mathcal{M}$ be the set of all positive Radon measures on $\mathbb{R}$. Although we focus on the real line, the interested reader should consult [9] for the general theory of random measures. Endow $\mathcal{M}$ with the topology $\mathcal{T}$ of vague convergence, that is, the weakest topology which makes the mappings

$$\mu \in \mathcal{M} \mapsto \mu f := \int_{\mathbb{R}} \mathrm{d}\mu\, f, \quad f \in C_K,$$

continuous. (Here, C_K is the set of continuous functions $f : \mathbb{R} \to \mathbb{R}$ with compact support.) A *random measure* is a random element of the space $(\mathcal{M}, \mathcal{T})$, viewed as a measurable space with σ-algebra generated by the sets in $\mathcal{T}$. A sequence $\lambda_1, \lambda_2, \ldots$ of random measures converges in distribution to a random measure λ if for any bounded continuous mapping $F : (\mathcal{M}, \mathcal{T}) \to \mathbb{R}$, we have $E[F(\lambda_i)] \to E[F(\lambda)]$ as $i \to \infty$. We write

$$\lambda_n \xrightarrow{(d)} \lambda,$$

to denote this notion. Convergence of λ_n to λ in distribution is equivalent to $\lambda_n f \xrightarrow{(d)} \lambda f$ for any continuous $f \in C_K$ (see Theorem 4.2 in [9]). The latter convergence is convergence in distribution of real-valued random variables.

Theorem 1 *There exists a random measure μ_∞ such that $\mu_n \xrightarrow{(d)} \mu_\infty$. Moreover, $\mu_\infty(\mathbb{R}) = 1$ a.s. The random probability measure μ_∞ almost surely admits a density $(L_a)_{a\in\mathbb{R}}$ with respect to the Lebesgue measure such that $a \mapsto L_a$ has compact support and is Hölder continuous with exponent $1/2 - \varepsilon$ for every $\varepsilon > 0$.*

We can think of μ_∞ as the occupation measure of the infinite iteration of i.i.d. Brownian motions. Thus, the random function $(L_a)_{a\in\mathbb{R}}$ must be thought of as the local time of this infinite iteration. The convergence of the last theorem will be obtained by proving convergence of finite-dimensional marginals of the iterated Brownian motions, namely:

Theorem 2 *For any integer $p \geq 1$, there exists a random vector $(X_i)_{1\leq i\leq p}$ such that for any pairwise distinct nonzero real numbers $x_1, x_2, \ldots, x_p$, we have the following convergence in distribution*:

$$\big(W_n(x_i)\big)_{1\leq i\leq p} \xrightarrow[n\to\infty]{(d)} (X_i)_{1\leq i\leq p}. \tag{2}$$

The random variables $X_1, \ldots, X_p$ and the differences $X_i - X_j$, $i \neq j$, have all identical distribution which is that of a "signed exponential" with parameter 2, i.e., a distribution with density $e^{-2|x|}$, $x \in \mathbb{R}$.

The case $p = 1$ has already been noticed in the Physics literature [15] in the context of infinite iteration of i.i.d. random walks. We are unfortunately unable to give an explicit formula for the distribution of $(X_1, \ldots, X_p)$ when $p \geq 2$; see Sect. 4 for a discussion and simulations of this intriguing probability distribution. It is quite interesting to observe that the marginals and differences have all the same distribution but are, of course, dependent. In a certain sense, the infinite iteration is both self-similar at all scales and long-range dependent.

The paper is organized as follows. The second section is devoted to the proof of Theorem 2, which is the cornerstone in the proof of Theorem 1. We then turn to the study of occupation measures in Sect. 3. The last section presents some open questions and comments.

2 Finite-Dimensional Marginals

In this section, we prove Theorem 2. The convergence of one-dimensional marginal is a special case because we can explicitly give its limiting distribution, which will be of great use throughout this paper. In the general case, the convergence of finite-dimensional marginals comes from ergodic property of random iterations of independent Brownian motions.

2.1 One-Dimensional Marginals

Let $\mathbb{R}^* := \mathbb{R} \setminus \{0\}$. For $\lambda > 0$, we denote by $\pm\mathcal{E}(\lambda)$ a signed exponential distribution with parameter λ, i.e., one which has density proportional to $e^{-\lambda|x|}$, $x \in \mathbb{R}$.

Proposition 3 *For any $t \in \mathbb{R}^*$, we have the following convergence in distribution*:

$$W_n(t) \xrightarrow[n\to\infty]{(d)} \pm\mathcal{E}(2). \tag{3}$$

Remark 1 Proposition 3 already implies that the sequence of processes (W_n) is not tight for the topology of uniform convergence on compact intervals. Indeed, if this were the case, and since $W_n(0) = 0$, we would have that $\sup_{n\geq 1} P(W_n(\varepsilon) > \eta) \to 0$ as $\varepsilon \to 0$ for every $\eta > 0$, and this contradicts Proposition 3.

Proof By standard properties of Gaussian variables we have the following chain of equalities in distribution:

$$\begin{aligned}
W_n(t) &= B_n\big(\cdots\big(B_3\big(B_2\big(B_1(t)\big)\big)\big)\cdots\big) \\
&\overset{(d)}{=} B_n\big(\cdots\big(B_3\big(B_2\big(\sqrt{|t|}B_1(1)\big)\big)\big)\cdots\big) \\
&\overset{(d)}{=} B_n\big(\cdots\big(B_3\big(|t|^{1/4}\sqrt{\big|B_1(1)\big|}B_2(1)\big)\big)\cdots\big) \\
&\overset{(d)}{=} \pm t^{2^{-n}} \prod_{i=0}^{n-1} |\mathcal{N}_i|^{2^{-i}},
\end{aligned}$$

where $\mathcal{N}_i$ are i.i.d. standard normal variables, and $\pm$ is an independent fair random sign. It is then easy to see that the absolute value of the right-hand side of the last display actually converges almost surely as $n \to \infty$. Indeed, the series of $P(|\log(|\mathcal{N}_i|^{2^{-i}})| > i^{-2})$ is summable, and an application of the Borel–Cantelli lemma proves the claim. Thus, if we set

$$\mathcal{X} := \lim_{n\to\infty} \prod_{i=0}^{n-1} |\mathcal{N}_i|^{2^{-i}} \quad \text{a.s.},$$

we have the convergence $W_n(t) \to \pm\mathcal{X}$ in distribution as $n \to \infty$, where $\pm$ is a fair random sign independent of $\mathcal{X}$. Notice that the limit does not depend on t as long as $t \neq 0$. To identify the limit distribution, we note that $\mathcal{X}$ satisfies the following recursive distributional equation:

$$\mathcal{X} \overset{(d)}{=} \left|\mathcal{N}(0, \mathcal{X})\right| \overset{(d)}{=} \sqrt{\mathcal{X}} \cdot |\mathcal{N}|, \tag{E}$$

where $\mathcal{N}$ is a normal variable independent of $\mathcal{X}$. Iterating this equation and applying the same arguments as above, it is easy to see that it admits a unique fixed point as long as $\mathcal{X} > 0$ almost surely. Guided by the result of [15], we verify that the exponential variable $\mathcal{E}_2$ of density $2e^{-2x}\mathbf{1}_{x>0}$ also satisfies (E). One way to see this is to identify moments. Indeed, recall that for every $\alpha > 0$, we have

$$E\big[|\mathcal{N}|^\alpha\big] = 2^{\alpha/2}\frac{\Gamma(\frac{p+1}{2})}{\sqrt{\pi}}, \qquad E\big[\mathcal{E}_2^\alpha\big] = 2^{-\alpha}\Gamma(\alpha+1),$$

where Γ is the standard Gamma function. Easy manipulations with the Γ function then imply that $E[|\mathcal{N}|^\alpha]E[\mathcal{E}_2^{\alpha/2}] = E[\mathcal{E}_2^\alpha]$ for every $\alpha \geq 0$. Since these variables are determined by their moments, we deduce that $\mathcal{E}_2 = \sqrt{\mathcal{E}_2}|\mathcal{N}|$ in distribution as desired. $\square$

2.2 General Case

The goal of this section is to prove Theorem 2 in the case $p \geq 2$. The convergence (2) will be achieved by applying arguments from the theory of Harris chains. The idea is to consider the random transformation which associates with any p points $x_1, \dots, x_p \in \mathbb{R}$ the images $B(x_1), \dots, B(x_p)$ of these p points under a two-sided Brownian motion B and to show that independent applications of this map possess an ergodic property. See Fig. 2. That is, for any initial state $(x_1, \dots, x_p)$, the distribution of $(W_n(x_1), \dots, W_n(x_p))$ converges weakly to a unique invariant probability measure.

Let $p \geq 2$. Denote by $\mathcal{R}^p$ the set of p-uplets $(x_1, \dots, x_p)$ of pairwise distinct nonzero real numbers. Note that if $\mathbf{x} = (x_1, \dots, x_p) \in \mathcal{R}^p$, then its image $B(\mathbf{x}) := (B(x_1), \dots, B(x_p))$ under a two-sided Brownian motion B almost surely belongs to $\mathcal{R}^p$.

Proposition 4 *For any $p \geq 2$, there is a unique probability measure ν_p on $\mathcal{R}^p$ such that if $(X_1, \dots, X_p)$ is distributed according to ν_p and if B is an independent two-sided Brownian motion, then we have the equality in distribution*

$$\big(B(X_i)\big)_{1\leq i\leq p} \overset{(d)}{=} (X_i)_{1\leq i\leq p}. \tag{4}$$

Furthermore, for any $\mathbf{x} = (x_1, \dots, x_p) \in \mathcal{R}^p$, we have the following convergence in distribution:

$$\big(W_n(x_i)\big)_{1\leq i\leq p} \xrightarrow[n\to\infty]{(d)} (X_i)_{1\leq i\leq p}.$$

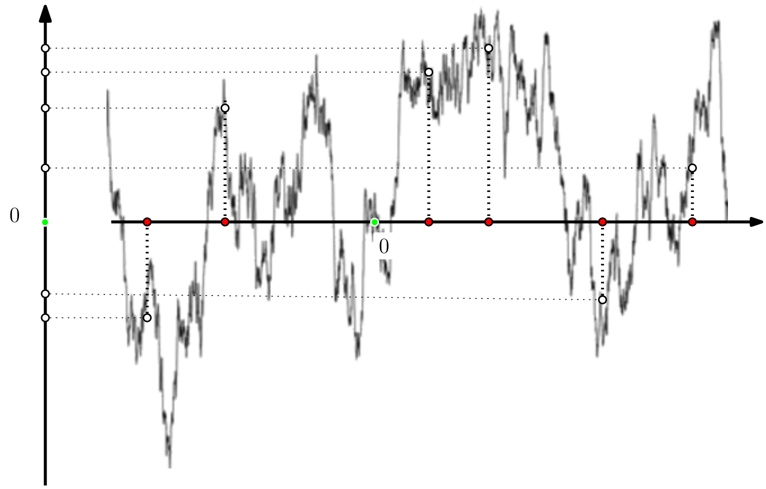

Fig. 2 A 6-uplet and its image after applying an independent Brownian motion

Notice that this argument does not give an explicit expression for the stationary probability measures ν_p but characterizes them uniquely by (4). Let $p \geq 2$ be a fixed integer. Fix also a p-uplet $\mathbf{x} = (x_1, \dots, x_p) \in \mathcal{R}^p$. For every $n \geq 1$, we set

$$\mathcal{W}_n := \big(W_n(x_1), \dots, W_n(x_p)\big).$$

Thus, the process $(\mathcal{W}_n)_{n\geq 0}$ is a Markov chain with state space $\mathcal{R}^p$, starting from $\mathcal{W}_0 := \mathbf{x}$, with transition probability kernel given by

$$\mathsf{P}(\mathbf{y}; A) = P\big[\big(B(y_1), \dots, B(y_p)\big) \in A\big]$$

for any $\mathbf{y} \in \mathcal{R}^p$ and $A \subset \mathcal{R}^p$, where B is a two-sided Brownian motion. We denote by $E_{\mathbf{x}}[\cdot]$ the expectation of this chain started from $\mathbf{x} \in \mathcal{R}^p$. Since we restricted ourselves to $\mathcal{R}^p$, the chain $(\mathcal{W}_n)$ is easily seen to be irreducible with respect to the p-dimensional Lebesgue measure on $\mathcal{R}^p$ and aperiodic. We will show that the chain is in fact positive Harris recurrent, which will imply that it admits a unique invariant probability measure, and Proposition 4 will directly follow from it, see [13, Theorem 13.0.1]. The key to prove this is to consider the following sets:

Definition 1 Fix $M > 1$ a (large) real number. We say that a p-uplet $\mathbf{x} = (x_1, \dots, x_p) \in \mathcal{R}^p$ is M-sparse if we have

$$M^{-1} \leq \sup_{1\leq i\leq p} |x_i| \leq M \quad \text{and} \quad \min_{1\leq i\neq j\leq p} |x_i - x_j| \geq M^{-1}.$$

The set of all M-sparse p-uplets in $\mathcal{R}^p$ is denoted by S_M.

The basic observation is that if $\mathbf{x} = (x_1, \dots, x_p)$ and $\mathbf{y} = (y_1, \dots, y_p)$ are two M-sparse p-uplets and if B is a two-sided Brownian motion, then the two random p-

uplets $(B(x_1), \ldots, B(x_p))$ and $(B(y_1), \ldots, B(y_p))$ are mutually absolutely continuous with Radon–Nikodým derivative bounded from below by a positive constant $c_{p,M}$ depending on M and p only. The reason is that the ratio of the two densities $f(x_1, \ldots, x_p)/f(y_1, \ldots, y_p)$ is a continuous function on the compact set S_M. In other words, there exist two (dependent) Brownian motions B and $\tilde{B}$ such that $B(x_1) = \tilde{B}(y_1), \ldots, B(x_p) = \tilde{B}(y_p)$ with probability at least $c_{p,M}$. Put it otherwise, if we fix $\mathbf{x} \in S_M$, then

$$\mathsf{P}(\mathbf{x}, A) \geq c_{p,M} \mathsf{P}(\mathbf{y}, A)$$

for all $\mathbf{y} \in S_M$ and all measurable $A \subset \mathcal{R}^p$. Thus, Ney's minorization condition holds, and, therefore, the set S_M is a petite set in the sense of [13, Chap. 5].

Proof of Proposition 4 In order to prove that $(\mathcal{W}_n)$ is positive Harris recurrent, we will show that, for some $M > 0$, the expected return time to the petite set S_M by the Markov chain $(\mathcal{W}_n)$ started from $\mathbf{x}$ is bounded above by a finite constant, *uniformly* over all $\mathbf{x} \in S_M$.

The technical tool that we use here is the so-called drift condition, see [13], for a Lyapunov or "potential" function $V : \mathcal{R}^p \to \mathbb{R}_+$. It is required that V be unbounded on $\mathcal{R}^p$. The drift of such a function is defined by

$$DV(x) := \mathsf{P}V(x) - V(x) = E_{\mathbf{x}}\big[V(\mathcal{W}_1)\big] - V(\mathbf{x}), \quad \mathbf{x} \in \mathcal{R}^p.$$

Thus, D is the generator of the Markov chain. We will show that there exists a Lyapunov function V such that, for some $0 < a, b < \infty$,

$$DV(x) \leq -a + b\mathbf{1}_{\mathbf{x} \in S_M} \tag{5}$$

for every $\mathbf{x} \in \mathcal{R}^p$. More precisely, we show that the preceding condition is satisfied for the function V defined by

$$V(x_1, \ldots, x_p) := \max_{1 \leq i \leq p} |x_i| + \sum_{0 \leq i < j \leq p} |x_i - x_j|^{-1/2}$$

and for some $a, b, M > 0$. In the last display, we use the notation $x_0 := 0$ to avoid extra terms in the definition of V. It is convenient to consider the terms $U(\mathbf{x}) := \max_{1 \leq i \leq p} |x_i|$ and $G(\mathbf{x}) := \sum_{0 \leq i < j \leq p} |x_i - x_j|^{-1/2}$ comprising V separately. For any $\lambda > 0$,

$$\mathsf{P}U(x) = E_{\mathbf{x}}\big[U(\mathcal{W}_1)\big] = E\Big[\max_{1 \leq i \leq p} \big|B(x_i)\big|\Big] = \sqrt{\lambda} E\Big[\max_{1 \leq i \leq p} \big|B(x_i/\lambda)\big|\Big].$$

Letting $\lambda = U(\mathbf{x})$, we thus have

$$\mathsf{P}U(\mathbf{x}) \leq C_1 \sqrt{U(\mathbf{x})},$$

where $C_1 = E[\max\{|B(t)| : -1 \leq t \leq 1\}]$. For the other term, we have, with $\mathcal{N}$ a standard normal variable,

$$\mathsf{P}G(\mathbf{x}) = E_{\mathbf{x}}\big[G(\mathcal{W}_1)\big] = \sum_{0 \leq i < j \leq p} E\Big[\big|B(x_i) - B(x_j)\big|^{-1/2}\Big]$$

$$= \sum_{0 \le i < j \le p} |x_i - x_j|^{-1/4} E\big[|\mathcal{N}|^{-1/2}\big] \le C_2\sqrt{G(\mathbf{x})},$$

where $C_2 = pE[|\mathcal{N}|^{-1/2}]$. Putting the terms together, we find

$$\mathsf{P}V(\mathbf{x}) = \mathsf{P}U(\mathbf{x}) + \mathsf{P}G(\mathbf{x}) \le C_1\sqrt{U(\mathbf{x})} + C_2\sqrt{G(\mathbf{x})} \le C_3\sqrt{U(\mathbf{x}) + G(\mathbf{x})} = \sqrt{V(\mathbf{x})},$$

where $C_3 = 2\max(C_1, C_2)$. Thus, for all $x \in \mathcal{R}^p$,

$$DV(\mathbf{x}) \le C_3\sqrt{V(\mathbf{x})} - V(\mathbf{x}).$$

If $\mathbf{x} \in S_M$, then $V(\mathbf{x}) \le M + (p+1)^2\sqrt{M} < \infty$. If $\mathbf{x} \notin S_M$, then $|x_i| > M$ for some i or $|x_i - x_j| < 1/M$ for some $0 \le i < j \le p$. In the first instance, $V(\mathbf{x}) > M$; in the second instance, $V(\mathbf{x}) > \sqrt{M}$. So, for $M > 1$, we have $V(\mathbf{x}) > \sqrt{M}$ for all $\mathbf{x} \notin S_M$. Let $M = \max(16C_3^4, 1)$. Then for $\mathbf{x} \notin S_M$, we have $\sqrt{V(\mathbf{x})} > 2C_3$, implying that $DV(\mathbf{x}) < -C_3\sqrt{V(\mathbf{x})} < -C_2M^{1/4}$. Thus, (5) holds. Theorem 13.0.1 in [13] now applies, showing positive Harris recurrence. This finishes the proof of the proposition. □

Remark 2 The proof actually shows that there are constants $C_4, C_5 > 0$ such that

$$DV(\mathbf{x}) \le -C_4\sqrt{V(\mathbf{x})}$$

when $V(\mathbf{x}) > C_5$, enabling one to establish a rate of convergence to the stationary distribution. We shall not pursue this further herein.

2.3 Properties of the ν_ps

We can think of ν_p as a measure on the product space $\mathbb{R}^p$ (by letting it have mass 0 outside $\mathcal{R}^p$). The probability measures $(\nu_p)_{p\ge1}$ are consistent. This follows from the fact that ν_p is the limit in distribution of $(W_n(x_1), \dots, W_n(x_p))$, whereas ν_{p+1} is the limit of $(W_n(x_1), \dots, W_n(x_p), W_n(x_{p+1}))$, for any $(x_1, \dots, x_p, x_{p+1}) \in \mathcal{R}^p$. Therefore, ν_p is the projection of ν_{p+1} on the first p coordinates. By Kolmogorov's extension theorem, there is a probability measure ν on $\mathbb{R}^{\mathbb{N}}$ such that ν_p is obtained from ν by projecting on the first p coordinates. The family $((\nu_p)_{p\ge1}, \nu)$ has some further properties.

2.3.1 Exchangeability

The last statement of Proposition 4 actually shows that ν is exchangeable, that is, invariant under permutations of finitely many coordinates. Let $(X_1, X_2, \dots)$ be a random element of $\mathbb{R}^{\mathbb{N}}$ with law ν. By the classical de Finetti/Ryll–Nardzewski/Hewitt–Savage theorem (Theorem 11.10 in [11]) we know that there exists a random element μ_∞ of $\mathcal{M}$ (that is, μ_∞ is a random probability measure) such that the law of $(X_1, X_2, \dots)$ conditionally on μ_∞ is a product measure (the law of i.i.d. random variables). The random measure μ_∞ will be identified and interpreted in the next section as the occupation measure of the infinitely iterated Brownian motion.

2.3.2 Stationarity of the Increments

The family of measures $(\nu_p)_{p\geq 1}$ also possesses another property which can be described as stationarity of the increments.

Proposition 5 *Let $p \geq 2$, and let $(X_1, X_2, \dots, X_p)$ be distributed according to ν_p. Then for every $1 \leq \ell \leq p$, we have*

$$(X_1 - X_\ell, \dots, X_{\ell-1} - X_\ell, X_{\ell+1} - X_\ell, \dots, X_p - X_\ell) \overset{(d)}{=} \nu_{p-1}.$$

Proof Let $(x_1, x_2, \dots, x_p) \in \mathcal{R}^p$. It is easy to prove by induction on $n \geq 1$ and using elementary manipulation of the Gaussian distribution that the random vector $(W_n(x_i) - W_n(x_\ell))_{i\neq\ell}$ has the same distribution as the vector $(W_n(x_i - x_\ell))_{i\neq\ell}$. Notice that the vector $(x_i - x_\ell)_{i\neq\ell} \in \mathcal{R}^{p-1}$, thus we can apply Proposition 4, and this finishes the proof of the proposition. □

3 The Occupation Measure

3.1 Existence of μ_∞

Recall the definition of the random measures μ_n given by formula (1) in the Introduction and the notion of convergence in distribution of random elements of $(\mathcal{M}, \mathcal{T})$. With Theorem 2 in our hands, it is now easy to prove convergence in distribution of the random measures μ_n. We first need a lemma, characterizing convergence in distribution of random elements of $(\mathcal{M}, \mathcal{T})$, tailor-made for our case. The lemma can be of independent interest.

Lemma 6 *Let $\lambda_1, \lambda_2, \dots$ be random probability measures on $\mathbb{R}$. The following are equivalent*:

(i) *The sequence λ_n converges in distribution to some random probability measure.*
(ii) *For each $n \geq 1$, and conditionally on λ_n, let $X_1^n, X_2^n, \dots$ be i.i.d. real-valued random variables each with* (*conditional*) *law λ_n*:

$$P\big(X_1^n \in A_1, \dots, X_p^n \in A_p \mid \lambda_n\big) = \lambda_n(A_1) \cdots \lambda_n(A_p) \quad a.s.$$

for all $p \geq 1$ and Borel sets $A_1, \dots, A_p$. The random vector $(X_1^n, \dots, X_p^n)$ converges in distribution as $n \to \infty$ to some probability measure on $\mathbb{R}^p$.

Proof The implication (i) $\Rightarrow$ (ii) is an easy consequence of [9, Theorem 4.2]. For the other direction, fix $f : \mathbb{R} \to \mathbb{R}$, $f \in C_K$. We will show that the random variable $\lambda_n f = \int_{\mathbb{R}} f \,\mathrm{d}\lambda_n$ converges in distribution as $n \to \infty$. Consider the random probability measure

$$\xi_n^{(p)} := p^{-1} \sum_{i=1}^{p} \delta_{X_i^n}$$

(i.e., the empirical distribution of $(X_1^n, \dots, X_p^n)$). We compare $\lambda_n f$ to $\xi_n^{(p)} f = p^{-1} \sum_{i=1}^p f(X_i^n)$. We have

$$P\big(\big|\lambda_n f - \xi_n^{(p)} f\big| \geq \varepsilon\big) = E\big[P\big(\big|\lambda_n f - \xi_n^{(p)} f\big| \geq \varepsilon \mid \lambda_n\big)\big] \leq \frac{\|f\|_\infty^2}{\varepsilon^2 p}$$

by Chebyshev's inequality. On the other hand, $\xi_n^{(p)} f$ converges in distribution as $n \to \infty$. Hence, $\lambda_n f$ converges in distribution, and thus λ_n converges in distribution to some random measure λ. It remains to prove that λ almost surely has mass one. Since X_1^n converges in distribution, it follows that the sequence of random variables $\{X_1^n\}_{n\geq 1}$ is tight. Thus, for every $\varepsilon > 0$, there exists $M > 0$ such that $P(|X_n^1| > M) \leq \varepsilon$ for every $n \geq 1$. Conditionally on λ_n, we have $P(|X_1^n| > M \mid \lambda_n) = 1 - \lambda_n([-M, M])$. Taking expectation, we deduce that $E[\lambda_n([-M, M])] \geq 1 - \varepsilon$ for every $n \geq 1$. This is sufficient to apply Theorem 4.9 in [10] and deduce that λ is almost surely a random probability measure. □

Let us go back to our setting and show that the occupation measures μ_n converge to a random probability measure μ_∞. Let $p \geq 1$, and, conditionally on μ_n, let $X_1^n, \dots, X_p^n$ be i.i.d. random variables with common distribution μ_n. We show that $(X_1^n, \dots, X_p^n)$ converges (unconditionally) to the random vector $(X_1, \dots, X_p)$ of law ν_p identified in Proposition 4. Indeed, by the definition of μ_n, for any Borel bounded function $f : \mathbb{R}^p \to \mathbb{R}$ we have, by Fubini's theorem,

$$\begin{aligned} E\big[f\big(X_1^n, \dots, X_p^n\big)\big] &= E\bigg[\int_{[0,1]^p} \mathrm{d}u_1 \cdots \mathrm{d}u_p \; f\big(W_n(u_1), \dots, W_n(u_p)\big)\bigg] \\ &= \int_{[0,1]^p} \mathrm{d}u_1 \cdots \mathrm{d}u_p \; E\big[f\big(W_n(u_1), \dots, W_n(u_p)\big)\big]. \end{aligned}$$

The last integral converges to $E[f(X_1, \dots, X_p)]$ as $n \to \infty$ because of Theorem 2 and dominated convergence. Applying Lemma 6, we get the existence of a random probability measure μ_∞ such that $\mu_n \to \mu_\infty$ in distribution as $n \to \infty$.

3.2 Support of μ_∞

Proposition 7 *Almost surely, the random probability measure μ_∞ has a bounded support.*

In order to prove the last proposition, we use a very general fact:

Lemma 8 *Let λ_n be a sequence of random probability measures converging in distribution to a random probability measure λ_∞. Suppose that the support of λ_n is contained in $[A_n, B_n]$ and that the sequences of random variables (A_n) and (B_n) are tight. Then λ_∞ has a compact support almost surely.*

Proof Let us argue by contradiction and suppose that λ_∞ has probability at least $\varepsilon > 0$ of having an unbounded support. By the assumption made on A_n and B_n

there exists $M > 0$ such that $P(|A_n| \geq M) \leq \varepsilon/10$ and $P(|B_n| \geq M) \leq \varepsilon/10$ for every $n \geq 1$. For the $M > 0$ chosen above, there exists $\delta > 0$ such that we have $\lambda_\infty(]-M, M[^c) > \delta$ with probability at least $\varepsilon/2$. Now choose $p \geq \delta^{-1}$. Suppose that, conditionally on λ_∞, the random variables $X_1^\infty, \dots, X_p^\infty$ are i.i.d. with common law λ_∞. We then have

$$\begin{aligned}
&P\Big(\sup_{1\leq i\leq p} \big|X_i^\infty\big| \geq M\Big) \\
&\quad \geq E\Big[P\Big(\sup_{1\leq i\leq p} \big|X_i^\infty\big| \geq M \;\big|\; \lambda_\infty\big(]-M, M[^c\big) > \delta\Big)\mathbf{1}_{\lambda_\infty(]-M,M[^c)>\delta}\Big] \\
&\quad \geq \frac{\varepsilon}{2}\big(1-(1-\delta)^p\big) \geq \varepsilon\big(1-e^{-1}\big)/2. \qquad (6)
\end{aligned}$$

On the other hand, if $X_1^n, \dots, X_p^n$ are i.i.d., conditionally on λ_n, we have

$$P\Big(\sup_{1\leq i\leq p} \big|X_i^n\big| \geq M\Big) \leq P\big(|A_n| \geq M\big) + P\big(|B_n| \geq M\big) \leq \varepsilon/5. \qquad (7)$$

Since for p fixed we have $(X_i^n)_{1\leq i\leq p} \to (X_i^\infty)_{1\leq i\leq p}$ in distribution as $n \to \infty$, comparing (6) and (7) leads to a contradiction. $\square$

We further establish a result on the limit of the oscillation of W_n on an interval as $n \to \infty$. Recall, that the oscillation of a function f on an interval J is defined by $\operatorname{osc}(f; J) := \sup_{s,t\in J} |f(t) - f(s)|$, and let, for $t > 0$,

$$\Delta_n(t) := \operatorname{osc}\big(W_n; [0, t]\big).$$

Lemma 9 *Let $D := \operatorname{osc}(B; [0, 1])$ be the oscillation of a Brownian motion on a unit interval, and let $D_0, D_1, D_2, \dots$ be i.i.d. copies of D. Then, for all $t > 0$,*

$$\Delta_n(t) \xrightarrow[n\to\infty]{(d)} \prod_{i=0}^{\infty} D_i^{2^{-i}}.$$

Proof Let

$$I_n(t) := \inf_{x\in[0,t]} W_n(x), \qquad S_n(t) := \sup_{x\in[0,t]} W_n(x).$$

Then $\Delta_n(t) = S_n(t) - I_n(t)$, and

$$\begin{aligned}
\Delta_{n+1}(t) &= \sup_{0\leq x\leq t} B_{n+1}\big(W_n(x)\big) - \inf_{0\leq x\leq t} B_{n+1}\big(W_n(x)\big) \\
&= \sup_{I_n(t)\leq u\leq S_n(t)} B_{n+1}(u) - \inf_{I_n(t)\leq u\leq S_n(t)} B_{n+1}(u) \\
&= \operatorname{osc}\big(B_{n+1}; \big[I_n(t), S_n(t)\big]\big) \\
&\overset{(d)}{=} \operatorname{osc}\big(B_{n+1}; \big[0, \Delta_n(t)\big]\big)
\end{aligned}$$

$$\stackrel{(d)}{=} \sqrt{\Delta_n(t)}\,\mathrm{osc}\big(B_{n+1}; [0, 1]\big).$$

Thus, iterating this equation, we get in the spirit of the proof of Proposition 3 the following equality in distribution:

$$\Delta_n(t) \stackrel{(d)}{=} t^{2^{-n}} \prod_{i=1}^{n} D_i^{2^{-(i-1)}}.$$

An argument similar to the one used in the proof of Proposition 3 shows that the right-hand side of the last display actually converges almost surely as $n \to \infty$ to the infinite product $\prod_{i=0}^{\infty} D_i^{2^{-i}}$. Hence, $\Delta_n(t)$ converges in distribution to the same random variable. □

Remark 3 Roughly speaking, the oscillation of the infinitely iterated Brownian motion is the same, in distribution, over any interval of any length.

Proof of Proposition 7 By Lemma 9, $\Delta_n(1)$ converges in distribution, and it is tight. Thus, the support of μ_n is contained in a compact interval whose endpoints are tight. Applying Lemma 8, we deduce that, almost surely, μ_∞ has a bounded support, as required. □

3.3 Density of μ_∞

This section is devoted to the analysis of the properties of the density of μ_∞. We will proceed in two steps. First, using standard technique of Fourier analysis for occupation densities (see, e.g., [2]), we will prove that μ_∞ almost surely has a density which is in $\mathbb{L}^2$. At the same time, we obtain some estimates about this Fourier transform. We will then use a very general result of Pitt [14] on local times to prove that this density is in fact continuous and even Hölder continuous with exponent $1/2 - \varepsilon$ for every $\varepsilon > 0$.

3.3.1 Harmonic Analysis of μ_∞

For $\xi \in \mathbb{R}$, let

$$\Phi(\xi) := \int_{\mathbb{R}} \mathrm{d}\mu_\infty(x) \exp(\mathrm{i}\xi x)$$

be the Fourier transform of the random probability measure μ_∞.

Proposition 10 *For any $\xi \in \mathbb{R}$, we have*

$$E\big[\big|\Phi(\xi)\big|^2\big] = \frac{4}{4+\xi^2}. \tag{8}$$

Proof By the definition of the Fourier transform $\Phi(\xi)$, we have

$$E\big[\big|\Phi(\xi)\big|^2\big] = E\bigg[\int_{\mathbb{R}^2} \mathrm{d}\mu_\infty(x)\,\mathrm{d}\mu_\infty(y)\, e^{i\xi(x-y)}\bigg].$$

By the convergences already established we have

$$E\left[\int_{\mathbb{R}^2} \mathrm{d}\mu_\infty(x)\,\mathrm{d}\mu_\infty(y)\,e^{i\xi(x-y)}\right] = \lim_{n\to\infty} E\big[\exp\big(i\xi\big(W_n(U_1) - W_n(U_2)\big)\big)\big],$$

where U_1, U_2 are two independent random variables uniformly distributed over [0, 1] and also independent of the sequence of Brownian motions $B_1, B_2, \ldots$. By stationarity of the increments (see the proof of Proposition 5), for any $s, t \in \mathbb{R}$, we have $W_n(t) - W_n(s) = W_n(t-s)$ in distribution, and thus,

$$\begin{aligned} E\left[\int_{\mathbb{R}^2} \mathrm{d}\mu_\infty(x)\,\mathrm{d}\mu_\infty(y)\,e^{i\xi(x-y)}\right] &= \lim_{n\to\infty} E\big[\exp\big(i\xi\big(W_n(U_1 - U_2)\big)\big)\big] \\ &= \lim_{n\to\infty} \iint_0^1 \mathrm{d}u\,\mathrm{d}v\,E\big[\exp\big(i\xi\,W_n(u-v)\big)\big]. \end{aligned}$$

Applying Proposition 3 and the dominated convergence theorem, we get that

$$E\left[\int_{\mathbb{R}^2} \mathrm{d}\mu_\infty(x)\,\mathrm{d}\mu_\infty(y)\,e^{i\xi(x-y)}\right] = E\big[\exp\big(i\xi\big[\pm\mathcal{E}(2)\big]\big)\big] = \frac{4}{4+\xi^2}.$$

$\square$

In particular, applying Fubini's theorem, we deduce from (8) that

$$E\big[\|\Phi\|_2^2\big] = \int_{\mathbb{R}} \left(\frac{4}{4+\xi^2}\right)^2 \mathrm{d}\xi < \infty$$

and thus that $\Phi \in \mathbb{L}^2$ almost surely. By standard results on Fourier transforms, this implies that, almost surely, μ_∞ has density $(L_a)_{a\in\mathbb{R}}$ with respect to the Lebesgue measure which is in $\mathbb{L}^2$. Notice that estimates (8) in fact give us a bit more; namely, for every $0 < s < 1/2$, we have

$$\|\Phi\|_{H^s} := \sqrt{\int_{\mathbb{R}} \mathrm{d}\xi\,\big(1+\xi^2\big)^s \big|\Phi(\xi)\big|^2} < \infty, \quad \text{a.s.}$$

Applying the standard Sobolev inequality, we deduce that for every $0 < s < 1/2$, we have

$$\|L\|_{\mathbb{L}^{2/(1-2s)}} \le C\|\Phi\|_{H^s},$$

where C is a universal constant; see Theorem 1.38 in [1]. Since s can be arbitrarily close to $1/2$ and since L is itself a density (thus, $L \in \mathbb{L}^1$ a.s.), we get that $L \in \mathbb{L}^q$ for any $1 \le q < \infty$ almost surely.

3.3.2 Continuity of the Density

The idea to prove the continuity of the density $(L_a)_{a\in\mathbb{R}}$ is to apply once again a Brownian motion "on top of μ_∞" and to use the general theory developed in [14].

Formally, if μ_∞ has a density $(L_a)_{a\in\mathbb{R}}$ (which is in every $\mathbb{L}^q$, $1 \le q < \infty$), we define the random measure $\widetilde{\mu}_\infty$ by

$$\widetilde{\mu}_\infty f := \int \mathrm{d}\mu_\infty(x)\ f\big(B(x)\big) = \int \mathrm{d}a\, L_a\ f\big(B(a)\big),$$

where B is an independent two-sided Brownian motion. Clearly, we have

$$\widetilde{\mu}_\infty \overset{(d)}{=} \mu_\infty$$

(equality in distribution). So it suffices to show that $\widetilde{\mu}_\infty$ has a continuous density. This follows from the following lemma.

Lemma 11 *Let $g : \mathbb{R} \to \mathbb{R}_+$ be the density of a probability measure supported on a compact interval $I \subset \mathbb{R}$ such that $g \in \mathbb{L}^q$ for every $1 \le q < \infty$. Let also B be a two-sided Brownian motion and consider the occupation measure of B with respect to the measure $g(t)\mathrm{d}t$ defined as follows*:

$$\mu f = \int_{\mathbb{R}} g(t)\,\mathrm{d}t\ f\big(B(t)\big)$$

for any Borel positive function f. Then the random measure μ almost surely has a density which is locally Hölder continuous of exponent $1/2 - \varepsilon$ for any $\varepsilon > 0$.

Proof Let $0 < \delta < 1$ and choose $q \ge 1$ such that $(1+\delta)q^* < 2$, where q^* is defined by $1/q + 1/q^* = 1$. Applying Hölder's inequality, we get that

$$\sup_{s\in I}\int_I \mathrm{d}t\ \frac{g(t)}{\sqrt{|t-s|}^{1+\delta}} \le \left(\int_I \mathrm{d}t\ \big|g(t)\big|^q\right)^{1/q} \sup_{s\in I}\left(\int_I \mathrm{d}t\ |t-s|^{-(1+\delta)q^*/2}\right)^{1/q^*} < \infty.$$

We can thus apply Theorem 4 of [14] and get that μ has a continuous density L over $\mathbb{R}$ which satisfies a local Hölder condition $|L(x) - L(y)| \le K|x-y|^{\delta'}$ for any $\delta' < \delta/2$. $\square$

4 Comments

Occupation over Other Measures In this work we considered the occupation measures μ_n of W_n over the time interval [0, 1], but the proofs apply in the more general setting of occupation measures of W_n over any probability measure on $\mathbb{R}$ which is not atomic. More precisely, if λ is a probability measure on $\mathbb{R}$ which has no atom, we define the occupation of W_n over λ by

$$\mu_n^{(\lambda)} f := \int_{\mathbb{R}} \lambda(\mathrm{d}t) f\big(W_n(t)\big)$$

for any Borel $f : \mathbb{R} \to \mathbb{R}_+$. Then the random probability measures $\mu_n^{(\lambda)}$ converge to μ_∞ in distribution as well.

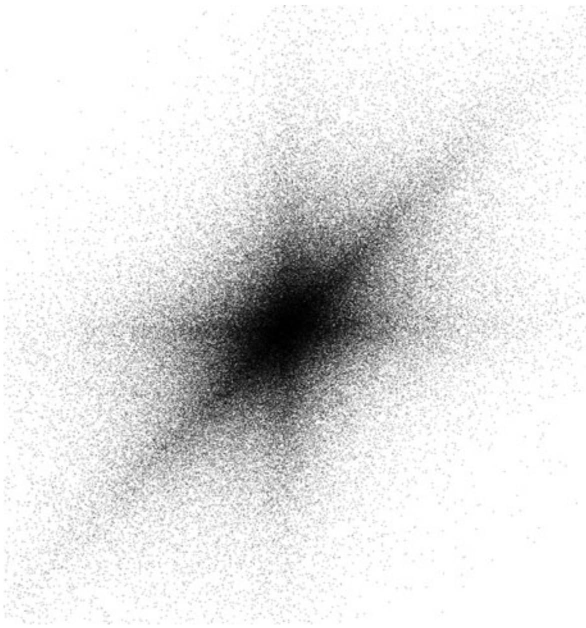

Fig. 3 About 300000 independent samples of the distribution ν_2. Simulation realized by Alexander Holroyd

Explicit Finite-Dimensional Marginals The consistent family distributions $\{\nu_p, p \geq 1\}$ introduced in Proposition 4 are the limiting distributions of the finite-dimensional marginals of the W_ns. They are characterized by (4). Although they arise naturally in the study of iteration of Brownian motions, to the best of our knowledge, they have not been investigated so far for $p \geq 2$, and in particular no explicit formulas are known for the density of ν_p, $p \geq 2$.

In Fig. 3 many independent points have been sampled according to ν_2. A clear shape "sea star" emerges, and we conjecture that the level-lines of the density of ν_2 would be dilatation of this unique shape. However, we do not have any candidate for this density.

Exponential Distribution The distributional equation (E) of Sect. 2.1 seems to be a new characterization of the exponential distribution. As such, it begs for a probabilistic explanation. This is unknown to us.

Ray–Knight Theorem In the spirit of the famous Ray–Knight theorem, do we have another way to describe the density of μ_∞ as a diffusion process?

Reflected Brownian Motion It is possible to extend part of the previous work to iterations of reflected Brownian motions. Namely, it is likely that Proposition 4 goes through but Proposition 5 fails, and thus the analysis of the Fourier transform of the limiting measure would be more challenging.

Fractional Brownian Motion Fractional Brownian motion generalizes Brownian motion in that it is a Gaussian H-self-similar process with stationary increments, where $0 < H < 1$. Again, it is likely that Proposition 4 goes through but the limit is even harder to describe. Of course, this raises the following interesting question: what kind of processes can be iterated ad libitum and result in some kind of limit?

Acknowledgements We are deeply indebted to Yuval Peres for insightful discussions.

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