Notes on Hermitian Matrices and Vector Spaces

1. Hermitian matrices

**Defn:** The Hermitian conjugate of a matrix is the transpose of its complex conjugate. So, for example, if

$$M = \begin{pmatrix} 1 & i \\ 0 & 2 \\ 1 - i & 1 + i \end{pmatrix},$$

then its Hermitian conjugate $M^\dagger$ is

$$M^\dagger = \begin{pmatrix} 1 & 0 & 1 + i \\ -i & 2 & 1 - i \end{pmatrix}.$$ 

In terms of matrix elements,

$$[M^\dagger]_{ij} = ([M]_{ji})^*.$$ 

Note that for any matrix

$$(A^\dagger)^\dagger = A.$$ 

Thus, the conjugate of the conjugate is the matrix itself.

**Defn:** A square matrix $M$ is said to be Hermitian (or self-adjoint) if it is equal to its own Hermitian conjugate, i.e.

$$M^\dagger = M.$$ 

For example, the following matrices are Hermitian:

$$\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{pmatrix}.$$ 

Note that a real symmetric matrix (the second example) is a special case of a Hermitian matrix.

**Theorem:** The Hermitian conjugate of the product of two matrices is the product of their conjugates taken in reverse order, i.e.

$$(AB)^\dagger = B^\dagger A^\dagger.$$ 

**Proof:**

$$[LHS]_{ij} = ([AB]_{ji})^* = (\sum_k [A]_{jk}[B]_{ki})^* = \sum_k ([A]_{jk})^*([B]_{ki})^*$$

$$= \sum_k ([B]_{ki})^*([A]_{jk})^* = \sum_k [B^\dagger]_{ik}[A^\dagger]_{kj} = [A^\dagger B^\dagger]_{ij} = [RHS]_{ij}.$$ 

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**Exercise:** Check this result explicitly for the matrices

\[
A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.
\]

**Theorem:** Hermitian Matrices have real eigenvalues.

**Proof**

\[Ax = \lambda x\]

so

\[x^\dagger Ax = \lambda x^\dagger x.\] \hspace{1cm} (1)

Take the complex conjugate of each side:

\[(x^\dagger Ax)^\dagger = \lambda^*(x^\dagger x)^\dagger.\]

Now use the last theorem about the product of matrices and the fact that \(A\) is Hermitian \((A^\dagger = A)\), giving

\[x^\dagger A^\dagger x = x^\dagger Ax = \lambda^*x^\dagger x.\] \hspace{1cm} (2)

Subtracting (1), (2), we have

\[(\lambda - \lambda^*)x^\dagger x = 0.\]

Since \(x^\dagger x \neq 0\), we have \(\lambda - \lambda^* = 0\), i.e. \(\lambda = \lambda^*\). **So \(\lambda\) is real.** (QED).

**Theorem:** Eigenvectors of Hermitian matrices corresponding to different eigenvalues are orthogonal.

**Proof**

Suppose \(x\) and \(y\) are eigenvectors of the hermitian matrix \(A\) corresponding to eigenvalues \(\lambda_1\) and \(\lambda_2\) (where \(\lambda_1 \neq \lambda_2\)). So we have

\[Ax = \lambda_1 x, \quad Ay = \lambda_2 y.\]

Hence

\[y^\dagger Ax = \lambda_1 y^\dagger x,\] \hspace{1cm} (3)

\[x^\dagger Ay = \lambda_2 x^\dagger y.\] \hspace{1cm} (4)

Taking the Hermitian conjugate of (4), we have

\[(x^\dagger Ay)^\dagger = y^\dagger Ax = \lambda_2^*(x^\dagger y)^* = \lambda_2 y^\dagger x,\]

where we have used the facts that \(A\) is Hermitian and that \(\lambda_2\) is real. So we have

\[y^\dagger Ax = \lambda_2 y^\dagger x.\] \hspace{1cm} (5)
Subtracting (5) from (3), we have
\[(\lambda_1 - \lambda_2)y^\dagger x = 0,\]
and since we are assuming \(\lambda_1 \neq \lambda_2\), we must have \(y^\dagger x = 0\), i.e. \(x\) and \(y\) are orthogonal. (QED).

**Example**
(i) Find the the eigenvalues and corresponding eigenvectors of the matrix
\[
A = \begin{pmatrix}
-1 & 0 & -2i \\
0 & 2 & 0 \\
2i & 0 & -1
\end{pmatrix}
\]
(ii) Put the eigenvectors \(X_1, X_2\) and \(X_3\) in *normalised* form (i.e. multiply them by a suitable scalar factor so that they have unit magnitude or ‘length’, \(X_i^\dagger X_i = 1\)) and check that they are orthogonal (\(X_i^\dagger X_j, \text{ for } j \neq i\)).
(iii) Form a 3 \(\times\) 3 matrix \(P\) whose columns are \(\{X_1, X_2, X_3\}\) and verify that the matrix 
\[
P^\dagger AP
\]
is diagonal. Do you recognise the values of the diagonal elements?

**Solution**
(i) Find the the eigenvalues in the usual way from the *characteristic* equation,
\[
\begin{vmatrix}
-1 - \lambda & 0 & -2i \\
0 & 2 - \lambda & 0 \\
2i & 0 & -1 - \lambda
\end{vmatrix} = 0.
\]
So,
\[(-1 - \lambda)(2 - \lambda)(-1 - \lambda) - 2i(-2i(2 - \lambda)) = 0\]
giving
\[(2 - \lambda)(\lambda^2 + \lambda - 3) = 0\]
which has solutions \(\lambda = 1, 2, -3\). These are the three eigenvalues of \(A\) (all real as expected).
To get the eigenvectors, go back to the eigenvalue equation
\[(A - \lambda I)X = 0.\]
Write this out for the three eigenvalues in turn:
For \(\lambda_1 = 1\),
\[
\begin{pmatrix}
-2 & 0 & -2i \\
0 & 1 & 0 \\
2i & 0 & -2
\end{pmatrix}
\begin{pmatrix}
x \\
y \\
z
\end{pmatrix} = 0.
\]
This matrix equation corresponds to 3 simple linear equations (of which only 2 are independent). We easily find
\[ z/x = i, y = 0. \]
So we can take
\[ X_1 = n_1 \begin{pmatrix} 1 \\ 0 \\ i \end{pmatrix} \]
where \( n_1 \) is a normalisation constant.
For \( \lambda_2 = 2 \),
\[ \begin{pmatrix} -3 & 0 & -2i \\ 0 & 0 & 0 \\ 2i & 0 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0. \]
This gives \( 3x = -2iz \) and \( 2ix = 3z \) so, \( x = z = 0 \) and \( y \) is arbitrary. So we can take
\[ X_2 = n_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}. \]
Finally, for \( \lambda_3 = -3 \),
\[ \begin{pmatrix} 2 & 0 & -2i \\ 0 & 5 & 0 \\ 2i & 0 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0. \]
From this we deduce \( x = iz \) and \( y = 0 \), so we can take
\[ X_3 = n_3 \begin{pmatrix} i \\ 0 \\ 1 \end{pmatrix}. \]
(ii) The norm or magnitude of \( X_1 \) is \( \sqrt{X_1^\dagger X_1} \). Now
\[ X_1^\dagger X_1 = n_1^2 \begin{pmatrix} 1 & 0 & -i \\ 0 & 0 & i \\ -i & i & 1 \end{pmatrix} = n_1^2(1 + 0 + 1) = 2n_1^2, \]
so we can take \( n_1 = 1/\sqrt{2} \) and get the normalised form of the first eigenvector
\[ X_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ i \end{pmatrix}. \]
Similarly we can take
\[ X_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad X_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} i \\ 0 \\ 1 \end{pmatrix}. \]
(iii) Form the matrix

\[ P = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & i \\ 0 & \sqrt{2} & 0 \\ i & 0 & 1 \end{pmatrix} \]

from the normalised eigenvectors. Then

\[
P^\dagger A P = \frac{1}{2} \begin{pmatrix} 1 & 0 & -i \\ 0 & \sqrt{2} & 0 \\ -i & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 & -2i \\ 0 & 2 & 0 \\ 2i & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 & i \\ 0 & \sqrt{2} & 0 \\ i & 0 & 1 \end{pmatrix}
\]

\[
= \frac{1}{2} \begin{pmatrix} 1 & 0 & -i \\ 0 & \sqrt{2} & 0 \\ -i & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -3i \\ 0 & 2\sqrt{2} & 0 \\ i & 0 & -3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -3 \end{pmatrix}
\]

Note that the corresponding eigenvalues appear as the diagonal elements. Note also that

\[ P^\dagger P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I. \]

A matrix satisfying this condition is said to be unitary.

2. Vector spaces

The vectors described above are actually simple examples of more general objects which live in something called a Vector Space. (*) Any objects which one can add, multiply by a scalar (complex number) and get a result of a similar type are also referred to as vectors.

Defn: A set of objects (vectors) \( x, y, \ldots \) is said to form a linear vector space \( \mathcal{V} \) if:

(i) The set is closed under addition (usual commutative and associative properties apply), i.e. if \( x, y \in \mathcal{V} \), then the ‘sum’ \( x + y \in \mathcal{V} \).

(ii) The set is closed under multiplication by a scalar, i.e. if \( \lambda \) is a scalar and \( x \in \mathcal{V} \), then \( \lambda x \in \mathcal{V} \). The usual associative and distributive properties must also apply.

(iii) There exists a null or zero vector \( \mathbf{0} \) such that \( x + \mathbf{0} = x \).

(iv) Multiplication by unity (\( \lambda = 1 \)) leaves the vector unchanged, i.e. \( 1 \times x = x \).

(v) Every vector \( x \) has a negative vector \( -x \) such that \( x + (-x) = 0 \).

[Note that, in the above, as is often the case, I have not bothered to denote vectors by bold face type. It is usually clear within the context, what is a vector and what is a scalar!]

(*) An excellent Mathematics text book covering all of the tools we need is the book ‘Mathematical Methods for Physics and Engineering’ by Riley, Hobson and Bence, CUP 1997. Chapter 7 contains material on vectors, vector spaces, matrices, eigenvalues etc. Chapter 15 has more on the application to functions which we describe in what follows.
It is immediately obvious that the usual 3-dimensional vectors form a vector space, as do the vectors acted upon by matrices as described above in the examples.

In the context of Quantum mechanics (see Mandl §1.1), we will assume that wave functions $\psi(x)$ form a vector space in the above sense. Multiplication of an ordinary vector by a matrix is a linear operation and results in another vector in the same vector space. Thus, if $x$ is a column vector with $n$ elements and $A$ is an $n \times n$ (complex) matrix, then $y = Ax$ is also a column vector of $n$ elements.

It is also clear that

$$A(x + y) = Ax + Ay \quad \text{and} \quad A(\lambda x) = \lambda (Ax)$$

so that multiplication by a matrix is seen to be a linear operation. A matrix is an example of what, in the general context of vector spaces, is called a linear operator.

Turning back to the space of wave functions, we note that the operation of differentiation $\partial / \partial x$ is a linear operation:

$$\frac{\partial}{\partial x}(\psi_1(x) + \psi_2(x)) = \frac{\partial \psi_1}{\partial x} + \frac{\partial \psi_2}{\partial x}.$$ 

**Inner products**

As with ordinary 3-D geometrical vectors, one can often define a inner or scalar or dot product. For $n$-dimensional complex vectors we define

$$\langle y | x \rangle \equiv y^\dagger x = \sum_{i=1}^{n} y_i^* x_i.$$ 

So the (magnitude)$^2$ of $x$ is

$$\langle x | x \rangle \equiv x^\dagger x = \sum_{i=1}^{n} |x_i|^2$$

and gives a measure of distance within the vector space. In the space of wave functions, an inner product can be defined by

$$\langle \psi_1 | \psi_2 \rangle \equiv \int_{-\infty}^{\infty} \psi_1(x)^* \psi_2(x) dx.$$  \hspace{1cm} (1)

Thus, the (norm)$^2$ or (magnitude)$^2$ of $\psi$ is given by

$$\langle \psi | \psi \rangle = \int_{-\infty}^{\infty} |\psi(x)|^2 dx.$$ 

It follows from the definition of the inner product (1) that

$$\langle \psi_1 | \psi_2 \rangle = \langle \psi_2 | \psi_1 \rangle^*.$$  \hspace{1cm} (2)
A vector space of this form, with an inner product, is sometimes referred to as a *Hilbert Space* (e.g. Mandl, Chapter 12).

**Properties of Hermitian linear operators**

We can now generalise the above Theorems about Hermitian (or self-adjoint) matrices, which act on ordinary vectors, to corresponding statements about Hermitian (or self-adjoint) linear operators which act in a Hilbert space, e.g. the space of wave functions in Quantum Mechanics.

For completeness, I rewrite the above Theorems (and Proofs) using the more general notation to describe scalar products. In this context, one often talks about eigenfunctions rather that eigenvectors if the ‘vectors’ happen to be functions or, specifically in our case, wave functions.

**Defn:** The *Hermitian* conjugate $A^\dagger$ of a linear operator $A$ is defined by

$$\langle y | Ax \rangle^* = \langle x | A^\dagger y \rangle .$$

An equivalent definition is given by

$$\langle Ax | y \rangle = \langle x | A^\dagger y \rangle .$$

This follows using property (2) above of the inner product.

Thus, if $A$ is a *Hermitian operator* ($A^\dagger = A$), we must have

$$\langle Ax | y \rangle = \langle x | Ay \rangle .$$

**Exercise:** Check that this definition agrees with that given when $A$ is a complex matrix.

**Theorem:** The Hermitian conjugate of the product of two linear operators is the product of their conjugates taken in reverse order, i.e.

$$(AB)^\dagger = B^\dagger A^\dagger .$$

**Proof:** For any two vectors $x$ and $y$,

$$\langle y | ABx \rangle = \langle (AB)^\dagger y | x \rangle .$$

But also

$$\langle y | ABx \rangle = \langle A^\dagger y | Bx \rangle = \langle B^\dagger A^\dagger y | x \rangle .$$

So we must have

$$(AB)^\dagger = B^\dagger A^\dagger .$$
Theorem: Hermitian (linear) operators have real eigenvalues.

Proof

\[ Ax = \lambda x \]

so

\[ \langle x | Ax \rangle = \lambda \langle x | x \rangle. \quad (1) \]

Take complex conjugate:

\[ \langle x | Ax \rangle^* = \lambda^* \langle x | x \rangle^* \]

so, using the definition of Hermitian conjugation (see above)

\[ \langle x | A^\dagger x \rangle = \lambda^* \langle x | x \rangle. \]

Now \( A \) is hermitian, so this means

\[ \langle x | Ax \rangle = \lambda^* \langle x | x \rangle. \quad (2) \]

Subtracting (1), (2), we have

\[ (\lambda - \lambda^*) \langle x | x \rangle = 0. \]

Since \( \langle x | x \rangle \neq 0 \), we have \( \lambda - \lambda^* = 0 \), i.e. \( \lambda = \lambda^* \). So \( \lambda \) is real.

Theorem: Eigenvectors (eigenfunctions) of Hermitian operators corresponding to different eigenvalues are orthogonal.

Proof

Suppose \( x \) and \( y \) are eigenvectors (eigenfunctions) of the hermitian operator \( A \) corresponding to eigenvalues \( \lambda_1 \) and \( \lambda_2 \) (where \( \lambda_1 \neq \lambda_2 \)). So we have

\[ Ax = \lambda_1 x, \quad Ay = \lambda_2 y. \]

Hence

\[ \langle y | Ax \rangle = \lambda_1 \langle y | x \rangle, \quad (3) \]
\[ \langle x | Ay \rangle = \lambda_2 \langle x | y \rangle. \quad (4) \]

Taking the complex conjugate of (4), we have

\[ \langle x | Ay \rangle^* = \langle y | A^\dagger x \rangle = \lambda_2^* \langle x | y \rangle^* = \lambda_2 \langle y | x \rangle, \]

where we have used the facts that \( \langle y | y \rangle^* = \langle y | x \rangle \), and that \( \lambda_2 \) is real. Then, since \( A \) is hermitian, it follows that

\[ \langle y | Ax \rangle = \lambda_2 \langle y | x \rangle. \quad (5) \]

Subtracting (5) from (3), we have

\[ (\lambda_1 - \lambda_2) \langle y | x \rangle = 0, \]
and since we are assuming $\lambda_1 \neq \lambda_2$, we must have $\langle y| x \rangle = 0$, i.e. $x$ and $y$ are orthogonal.

Example
The linear operator

$$K = -\frac{d^2}{dx^2}$$

acts on the vector space of functions of the type $f(x) (-L \leq x \leq L)$ such that

$$f(-L) = f(L) = 0.$$

(i) Show that $K$ is Hermitian.
(ii) Find its eigenvalues and eigenfunctions.

Solution
(i) We need to check if

$$\langle g| Kf \rangle = \langle f| Kg \rangle^*,$$

for any functions $f$ and $g$ in the given space. We have

$$LHS = \langle g| Kf \rangle = -\int_{-L}^{L} g^{*} \frac{d^2f}{dx^2} dx = [g^{*} \frac{df}{dx}]_{-L}^{L} + \int_{-L}^{L} \frac{dg^{*}}{dx} \frac{df}{dx} dx$$

$$= \int_{-L}^{L} \frac{dg^{*}}{dx} \frac{df}{dx} dx$$

where we have used integration by parts. Similarly

$$RHS^* = \langle f| Kg \rangle = -\int_{-L}^{L} f^{*} \frac{d^2g}{dx^2} dx = [f^{*} \frac{dg}{dx}]_{-L}^{L} + \int_{-L}^{L} \frac{df^{*}}{dx} \frac{dg}{dx} dx$$

$$= \int_{-L}^{L} \frac{df^{*}}{dx} \frac{dg}{dx} dx$$

$$= LHS^*. $$

Hence we conclude $K$ is Hermitian.

(ii) We require to solve the differential equation

$$Kf = \lambda f, \text{ i.e. } -\frac{d^2f}{dx^2} = \lambda f,$$

subject to the boundary conditions $f(-L) = f(L) = 0$. The eigenvalue equation is therefore

$$\frac{d^2f}{dx^2} = -\lambda f$$

which has the general solution (simple harmonic oscillator!)

$$f(x) = A \cos(\sqrt{\lambda} x) + B \sin(\sqrt{\lambda} x).$$
The boundary conditions then imply
\[
\begin{align*}
  f(L) &= 0 \Rightarrow A \cos(\sqrt{\lambda}L) + B \sin(\sqrt{\lambda}L) = 0 \\
  f(-L) &= 0 \Rightarrow A \cos(\sqrt{\lambda}L) - B \sin(\sqrt{\lambda}L) = 0.
\end{align*}
\]

There are two types of eigenfunctions

\textit{odd}: \( A = 0 \) and \( \sin(\sqrt{\lambda}L) = 0 \), so \( \sqrt{\lambda}L = n\pi \), \( (n = 0, 1, 2 \ldots) \)

giving eigenvalues and eigenfunctions of \( K \) as
\[
\lambda = \frac{n^2 \pi^2}{L^2}, \quad f^{(o)}_n(x) = B_n \sin\left(\frac{n\pi x}{L}\right), \quad (n = 0, 1, 2 \ldots)
\]

\textit{even}: \( B = 0 \) and \( \cos(\sqrt{\lambda}L) = 0 \), so \( \sqrt{\lambda}L = (n + 1/2)\pi \), \( (n = 0, 1, 2 \ldots) \)

giving eigenvalues and eigenfunctions of \( K \) as
\[
\lambda = \frac{(n + 1/2)^2 \pi^2}{L^2}, \quad f^{(e)}_n(x) = A_n \cos\left(\frac{(n + 1/2)\pi x}{L}\right), \quad (n = 0, 1, 2 \ldots)
\]

If you are feeling really keen, you could check that all these eigenfunctions are indeed orthogonal, i.e.
\[
\langle f^{(p)}_n | f^{(q)}_m \rangle = 0
\]
whenever \( m \neq n \), for \( p = o, e \) and \( q = o, e \).

\textbf{Basis vectors} In dealing with any vector space \( V \), it is very convenient to choose a set of \textit{basis vectors} in terms of which \textit{any} of the vectors in \( V \) can be written. The number of these needed is called the \textit{dimension} of the vectors space (maximum number of \textit{linearly independent vectors} in \( V \)). (Recall the discussion of such things in MATH102!)

For example, in ordinary 3D (Euclidean) space, we are all used to adopting \( \mathbf{i}, \mathbf{j} \) and \( \mathbf{k} \) as \textit{basis vectors}. There are, of course, 3 of them and they are clearly linearly independent. Actually, they are \textit{orthogonal} and \textit{unit} (\( \equiv \text{orthonormal} \)). We don’t have to use these as a basis. They don’t even have to be orthogonal! For example we could change basis to
\[
e_1 = \mathbf{i} + \mathbf{j}, \quad e_2 = \mathbf{i} - \mathbf{j}, \quad e_3 = \mathbf{i} + \mathbf{j} + \mathbf{k}.
\]

The important thing about a basis is that an arbitrary vector can always be written as a \textit{linear combination} of the basis vectors. Thus if \( \psi \in V \) and \( \{\phi_i\}, i = 1, 2 \ldots N \) is a basis of \( V \), we can write
\[
\psi = \sum_{i=1}^{N} c_i \phi_i.
\]
It is often convenient to choose the basis vectors to be the eigenvectors of some linear operator which acts on the space. In Quantum Mechanics, we will do this all the time!

**Example**

The matrix

\[ A = \begin{pmatrix} -1 & 0 & -2i \\ 0 & 2 & 0 \\ 2i & 0 & -1 \end{pmatrix} \]

(see page 3) acts on the 3D space of column vectors of the form

\[ \begin{pmatrix} x \\ y \\ z \end{pmatrix} . \]

Construct a basis for this 3D space consisting of the eigenvectors of \( A \). How do you know the 3 vectors are linearly independent? Express the vector

\[ v = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \]

in terms of these basis vectors.

**Solution**

See example on page 3, for the extraction of eigenvectors of \( A \):

\[ X_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ i \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad X_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} i \\ 0 \\ 1 \end{pmatrix} . \]

Since there are 3 of them and they are linearly independent, they form a basis. We know they are linearly independent because they are orthogonal (correspond to different eigenvalues). Now let

\[ v = \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \sum_{i=1}^{3} c_i X_i = c_1 \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ i \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c_3 \frac{1}{\sqrt{2}} \begin{pmatrix} i \\ 0 \\ 1 \end{pmatrix} . \]

There are two ways to solve this

(a) Equate components on each side of this vector equation, so finding 3 linear simultaneous equations in \( c_1, c_2 \) and \( c_3 \). Solve for these.

(b) Use the fact that the \( X_i \) are orthonormal to show that

\[ c_j = X_j^\dagger v . \]
Try (a) as an exercise. Now look at method (b) in more detail. We want

\[ v = \sum_{i=1}^{3} c_i X_i, \]

where

\[ X_j^\dagger X_i = \delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases} \]

Left multiply both sides of the equation for \( v \) by \( X_j^\dagger \):

\[ X_j^\dagger v = \sum_{i=1}^{3} c_i X_j^\dagger X_i = \sum_{i=1}^{3} c_i \delta_{ij} = c_j. \]

Thus we can calculate, for example

\[ c_1 = X_1^\dagger v = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & -i \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \frac{1}{\sqrt{2}} (a - ic). \]

Similarly, we find

\[ c_2 = b, \quad c_3 = \frac{1}{\sqrt{2}} (c - ia). \]