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NIM and related games

Winning Strategies

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Classification of Games

- *Combinatorial Games*
  - two players, alternating
  - finite set of positions, with rules on how to move between them
  - game has to end when one player can no longer move

- *Partisan Games*
  - Each player only is allowed to use his own resources (e.g. Chess)

- *Impartial Games*
  - All players have full access to all resources on their move
  - Allowed moves depend only on the current state of the game

Winning Strategy

Our goal is to find winning strategies for a large class of games.

“Winner” usually means “the last person to make a legal move”,

but we can always reverse that. This is not a new game, but we call it the “Misere” version of the original game.

Let’s do a warmup!

Play Takeaway 10

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You have a pile of 10 matchsticks

- Players alternate their moves.
- On each turn, the player may remove either 1, 2 or 3 sticks.
- The player taking the last matchstick wins.

Play 4 games in pairs and tell me if which player won.

Can you see a winning strategy?

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Winning Strategy

If you can leave your opponent a multiple of 4 matches, then you can always win from there.

Since the first player can take 2 on the first move, player 1 has a winning strategy.

Definitions

- A *P-Position* will be a guaranteed win for the Previous player
- An *N-Position* will be a guaranteed win for the Next player
- A *T-position* is a Terminal position with no possible moves left

Our starting position of 10 sticks was an N-position.

All positions with 0, 4, or 8 sticks were P-positions.

We can find all P and N positions

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1. We work backwards from the T-state. It is clearly P: the previous player has just made the last move and won the game.

2. Now take all possible moves that could have reached a P state, and label them with N.

3. Any position that can only reach N positions gets a label P

4. Repeat steps 2 and 3 until all positions are labeled.

Takeaway 10 fully analysed

0 1 2 3 4 5 6 7 8 9 10

0 1 2 3 4 5 6 7 8 9 10

0 1 2 3 4 5 6 7 8 9 10

0 1 2 3 4 5 6 7 8 9 10

0 1 2 3 4 5 6 7 8 9 10

0 1 2 3 4 5 6 7 8 9 10

0 1 2 3 4 5 6 7 8 9 10

Now a more complicated game: NIM

- There are three heaps of matches. Each heap can have any finite number of matches.
- Players can take any number of matches from any one of the heaps.
- Players must take at least one match.
- The player who takes the last match wins.

What is a winning strategy?

Play with heaps of size (5, 3, 2) and see if you can figure it out.

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Some observations

Any position with two equal heaps remaining is a definite win for the previous player (P).

Someone has to take the last match in a heap when two other heaps still have elements.

Let's do a full analysis. Remember, the P positions are what you want to achieve on your move to guarantee a win for yourself.

(5,3,2) NIM fully analysed

(0,0,0) (As the heaps are equivalent we can sort by size)

(1,0,0) (2,0,0) (3,0,0) (4,0,0) (5,0,0)

(1,1,0)

(2,1,0) (3,1,0) (4,1,0) (5,1,0) (1,1,1) (2,1,1) (3,1,1) (4,1,1) (5,1,1)

(2,2,0)

(3,2,0) (4,2,0) (5,2,0) (2,2,1) (2,2,2) (3,2,2) (4,2,2) (5,2,2)

(3,3,0)

(4,3,0) (5,3,0) (3,3,1) (3,3,2)

(3,2,1)

(4,2,1) (5,2,1) (4,3,1) (5,3,1) (4,3,2),(5,3,2)

What about (3,2,1) ?

It's clearly P. But building up these tables can be quite time consuming.

Is there a better way? Yes, but we need to lay some more groundwork:

Binary numbers and XOR operations.

Reminder: Binary numbers

$25 = 2*10^1 + 5*10^0$

$25 = 16+8+1 = 1*2^4+1*2^3+0*2^2+0*2^1+1*2^0$ = b11001 = b00011001

0 = b0	8 = b1000 etc.
1 = b1	9 = b1001
2 = b10	10 = b1010
3 = b11	11 = b1011
4 = b100	12 = b1100
5 = b101	13 = b1101
6 = b110	14 = b1110
7 = b111	15 = b1111

NIM-sums

We first write the NIM heaps in binary.

The NIM sum is the XOR (exclusive or, $\oplus$) of all three heap numbers.

A	**1**	**1**	**0**	**0**
B	**1**	**0**	**1**	**0**
A $\oplus$ B	**0**	**1**	**1**	**0**

(5,3,2) = (b101, b011, b010)   5$\oplus$3 = b101 $\oplus$ b011 = b110 = 6

Now 5$\oplus$3$\oplus$2 = 6$\oplus$2 = b110 $\oplus$b010 = b100 = 4

Some useful properties: a$\oplus$a=0   a$\oplus$0=a

Exercise: Work out the NIM sums of

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(5,2,1)

(3,3,0)

(4,2,1)

(3,2,1)

A	**1**	**1**	**0**	**0**
B	**1**	**0**	**1**	**0**
A ⊕ B	**0**	**1**	**1**	**0**

0 = b0
1 = b1
2 = b10
3 = b11
4 = b100
5 = b101
6 = b110
7 = b111
8 = b1000
9 = b1001
10 = b1010
11 = b1011
12 = b1100
13 = b1101
14 = b1110
15 = b1111

Some useful properties:

$a \oplus a = 0$ $a \oplus 0 = a$

Exercise: Work out the NIM sums of

(5,2,1) $b101 \oplus b010 \oplus b001 = b110 = 6$

(3,3,0) $b011 \oplus b011 \oplus b000 = b000 = 0$

(4,2,1) $b100 \oplus b010 \oplus b001 = b111 = 7$

(3,2,1) $b011 \oplus b010 \oplus b001 = b000 = 0$

(5,3,2) NIM fully analysed: NIM-sums

(0,0,0)0

(1,0,0)1 (2,0,0)2 (3,0,0)3 (4,0,0)4 (5,0,0)5

(1,1,0)0

(2,1,0)3 (3,1,0)2 (4,1,0)5 (5,1,0)4 (1,1,1)1 (2,1,1)2 (3,1,1)3 (4,1,1)4 (5,1,1)5

(2,2,0)0

(3,2,0)1 (4,2,0)6 (5,2,0)7 (2,2,1)1 (2,2,2)2 (3,2,2)3 (4,2,2)4 (5,2,2)5

(3,3,0)0

(4,3,0)7 (5,3,0)6 (3,3,1)1 (3,3,2)2

(3,2,1)0

(4,2,1)7 (5,2,1)6 (4,3,1)6 (5,3,1)7 (4,3,2)5 (5,3,2)4

Theorem

A position in a game of NIM is a P-position iff its NIM-sum equals 0.

To prove it we first prove two Lemmas:

Lemma 1: If a position had Nim-sum zero then the next move will make it non-zero.

Proof: Call (x_1,x_2,x_3) the position before the move with NIM-sum $s=0$ and (y_1,y_2,y_3) the position after the move with NIM-sum t.

The move has to subtract $x_k-y_k>0$ from heap k.

Then the new NIM-sum is $t = s \oplus (y_k-x_k) = 0 \oplus (y_k-x_k) \neq 0$ □

Lemma 2: If the position has a non-zero NIM-sum, the next move can always make it zero.

Again, we call (x_1,x_2,x_3) the position before the move with NIM-sum *s* and (y_1,y_2,y_3) the position after the move with NIM-sum *t*.

Proof: find the most significant bit in *s* and pick a heap which also has that bit set. At least one such heap must exist. Let that heap be k with x_k elements. Make it $y_k = s \oplus x_k <= x_k$ by removing $x_k - y_k > 0$ elements from the heap.

Then $t = s \oplus (s \oplus x_k - x_k) = s \oplus (s \oplus x_k \oplus x_k) = (s \oplus s) \oplus (x_k \oplus x_k) = 0 \oplus 0 = 0$

□

Prove the Theorem:

If you start with a NIM-sum 0 (P-position) then your opponent must make it non-zero (N-position) by Lemma 1 and you can turn it back to zero by Lemma 2.

Since each move will remove at least one element, you will eventually leave a T-position position with NIM-sum 0, meaning you have won on your last move.

□

If the game starts with a NIM sum of zero, the first player will have a winning strategy, else the second player has a winning strategy.

Winning strategy for Misere version:

Play with the same strategy as the full game.

Eventually you have a position that has only heaps with 0 or 1 elements left. In normal play you'd leave an even number of these, in misere play you'd leave an off number of these piles, thus your opponent has to take the last element and again, you win.

eg P1 leaves (3,2,1) P2 (3,1,1) P1 (1,1,1) P2 (1,1,0) P1 (1,0,0) P2 has to take the last element and P1 wins.

Generalisations

This will work with any number of heaps (n-heap NIM).

If you start with a NIM sum of zero, you will have a winning strategy, else your opponent has a winning strategy.

But many other impartial games also can be reduced to NIM!

Northcott's game

Not an impartial game!

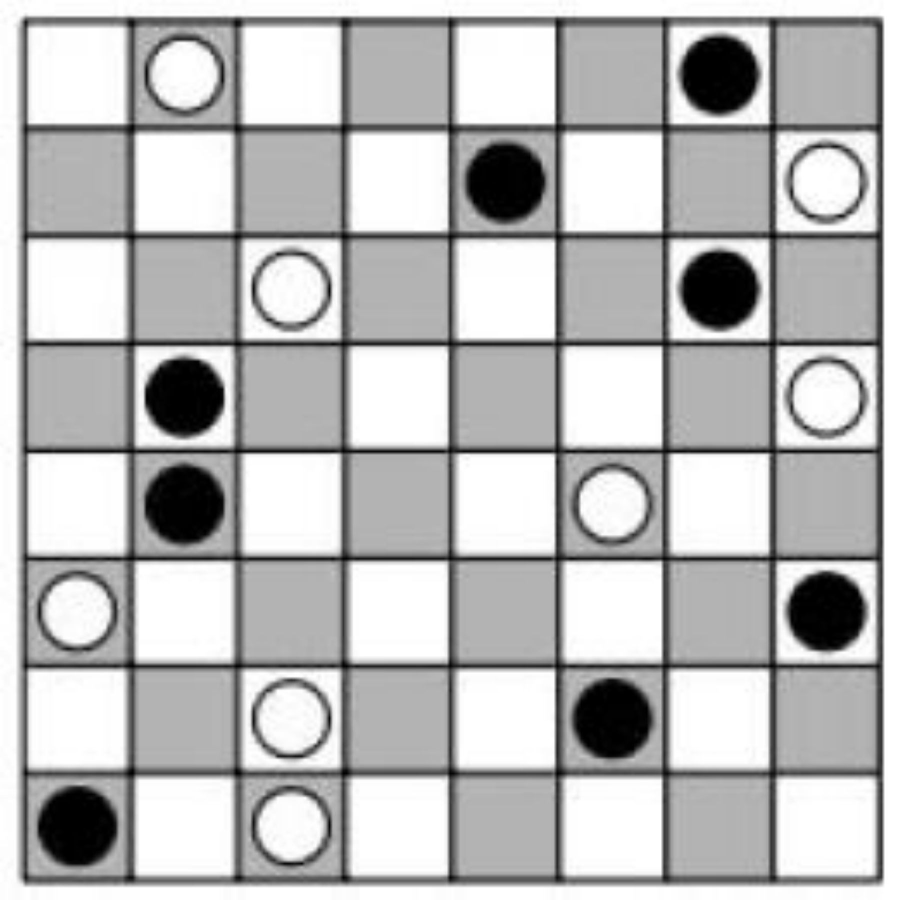
Each row on a checkerboard has one white and one black token.

Each player plays with their own colour.

On each move a player can move one of his tokens to the right but no further than the last square or until it bumps against another token.

The last person to make a move wins.

What is the winning strategy?



Northcott's game

Treat the number of open squares (n) between two tokens as a NIM pile with size n. You now have an 8-pile game of NIM, but the piles can also increase.

Play regular NIM, but each time your opponent *increases* a gap, *decrease* it by the same amount. The position then has the same NIM-sum, but you will be closer to the end of the game.

Eventually all of the rightmost squares will have tokens, and you can simply play your winning NIM strategy.

Nimble

Your board consists of boxes labeled 0, 1, 2, 3... with a finite number of coins placed in the boxes. Each box can contain any number of coins.

A move consists of taking at least one coin from one box and moving it to a box with a lower number.

The player who places the last coin in box 0 wins.

Strategy?

Treat each coin in box n as a NIM-pile of size n.

Summary

We have looked at impartial NIM-type games and found a winning strategy for them all.

Try to find other impartial games and see how they are like NIM.

For further reading, you may look up the Sprague-Grundy theorem, which generalizes our way of finding N and P positions.

I used material from an MIT Open course (ES.268)

https://ocw.mit.edu/courses