Talking about

Pick's Theorem on the area of polygons

whose vertices lie at 'lattice points' (points with whole number coordinates)

University of Liverpool
Maths Club
October 2020
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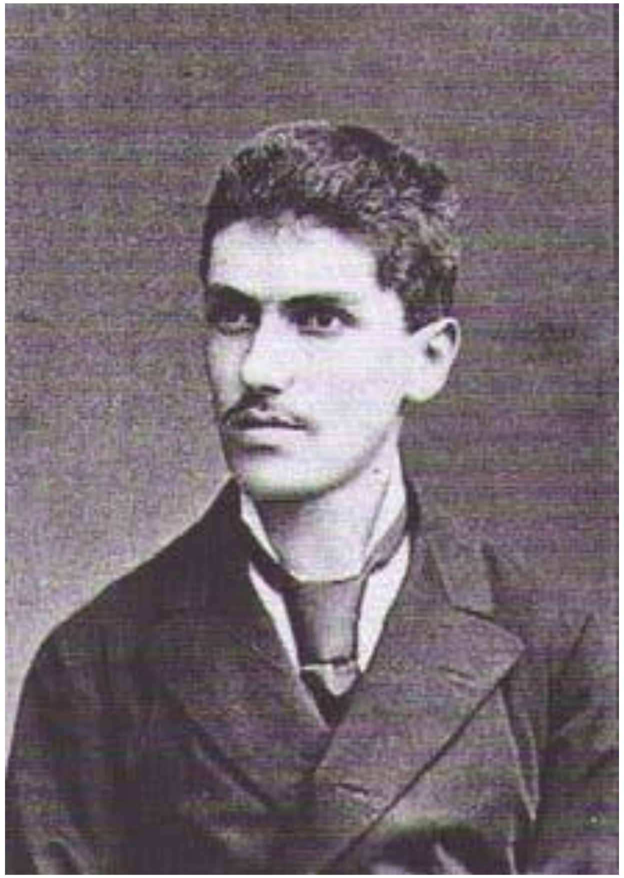
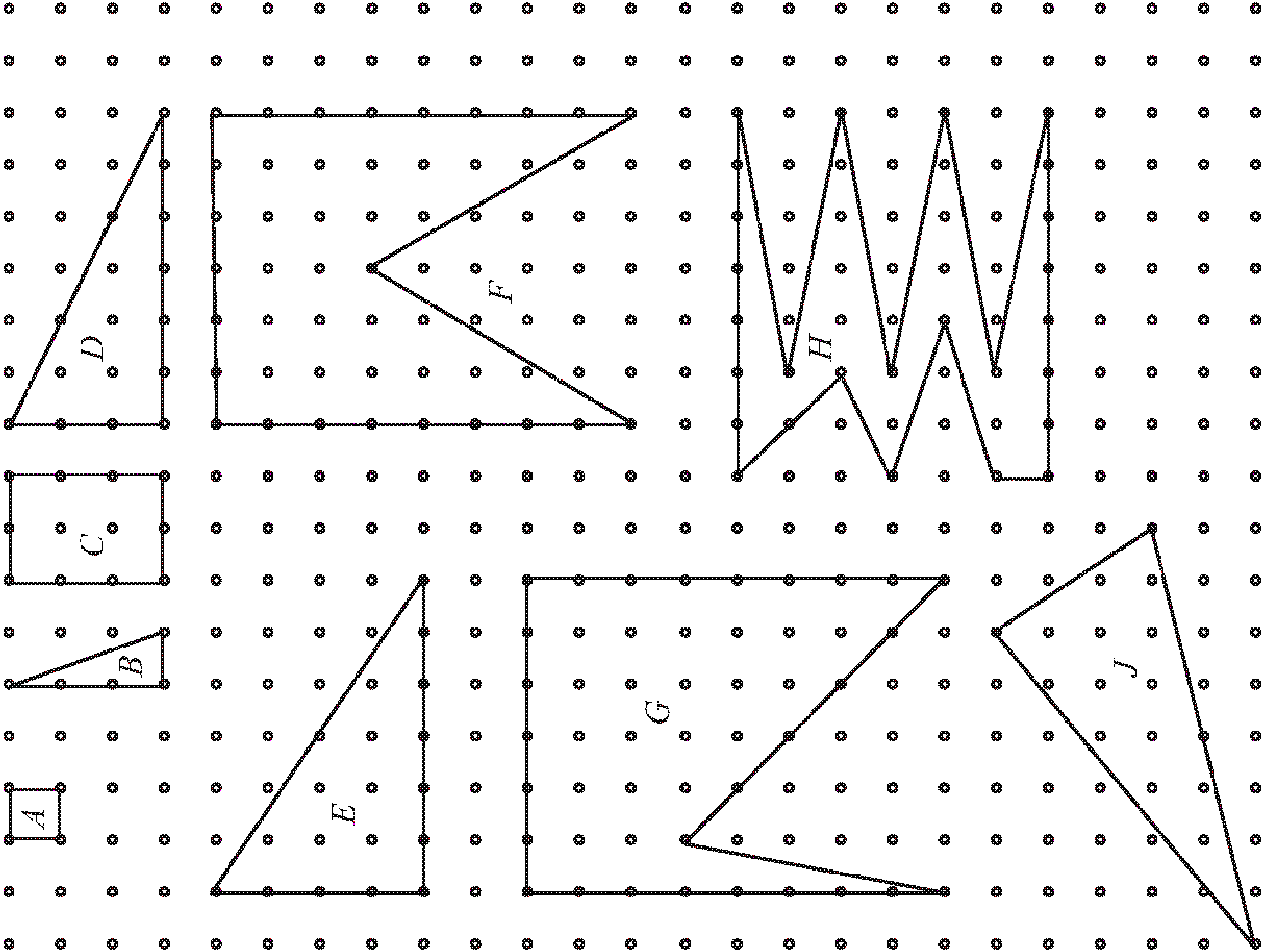


Georg Alexander Pick
1859-1942
(died in
Theresienstadt
concentration camp)

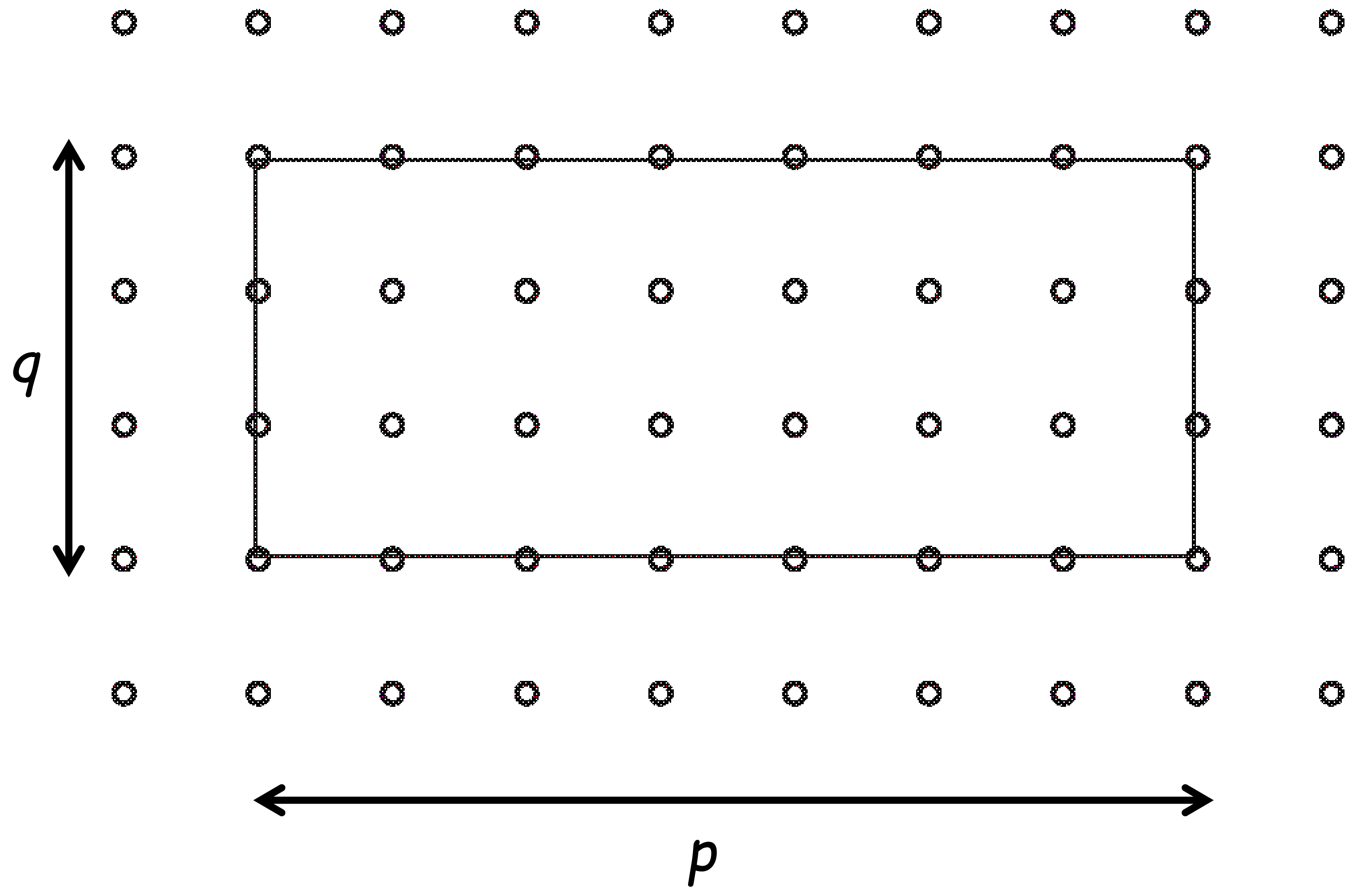


Polygons whose **vertices** lie on the ***lattice points*** (think of them as points with whole number coordinates)

What are the areas here?

	A	B	C	D	E	F	G	H	J
Area									
a									
b									
$\frac{1}{2}b$									

a = number of lattice points INSIDE, not on edges
b = number of lattice points on the edges

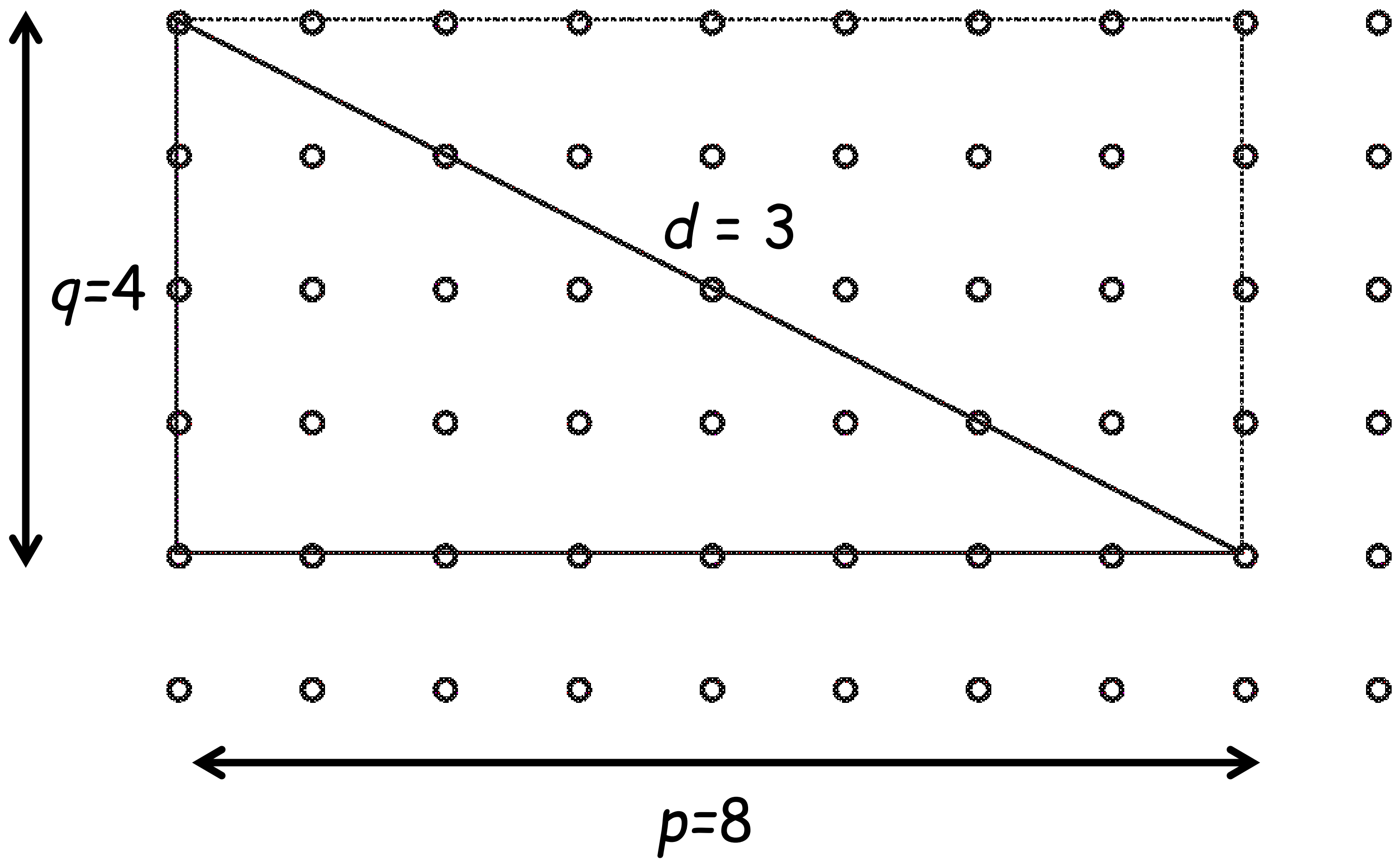


What is the count of lattice points inside and on the edges of a $p \times q$ rectangle?

Note *p*, *q* are the <u>lengths</u> of the sides.

a = number of lattice points INSIDE, not on edges
b = number of lattice points on the edges
area of rectangle = *pq* and in terms of *a*, *b* is ??

a or *A*: lattice points INSIDE
b or *B*: lattice points on the edges
d: points on the diagonal not at the ends

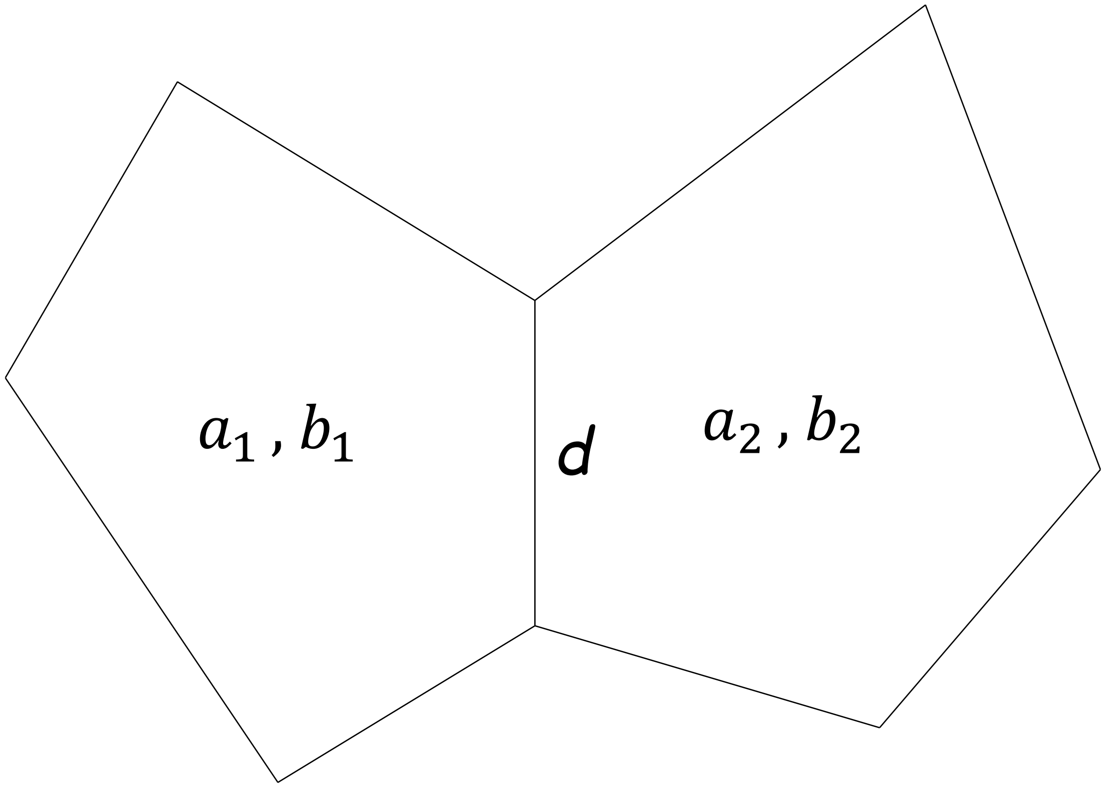


What are the numbers of lattice points on and inside this triangle? What about a right-angled triangle as half of any $p \times q$ rectangle?

A, *B* for the rectangle
a, *b* for the triangle
d lattice points on the diagonal but not at ends

(i) $A = 2a + d$
(ii) $B = 2b - 2d - 2$
(iii) $\frac{1}{2}(A + \frac{1}{2}B - 1) = a + \frac{1}{2}b - 1$

and the area formula works here too!



a_1, a_2 = lattice points inside
b_1, b_2 = lattice points on the edges for the separate shapes

d points on common line, omitting endpoints.

a, *b* the inside/edge numbers for the whole shape

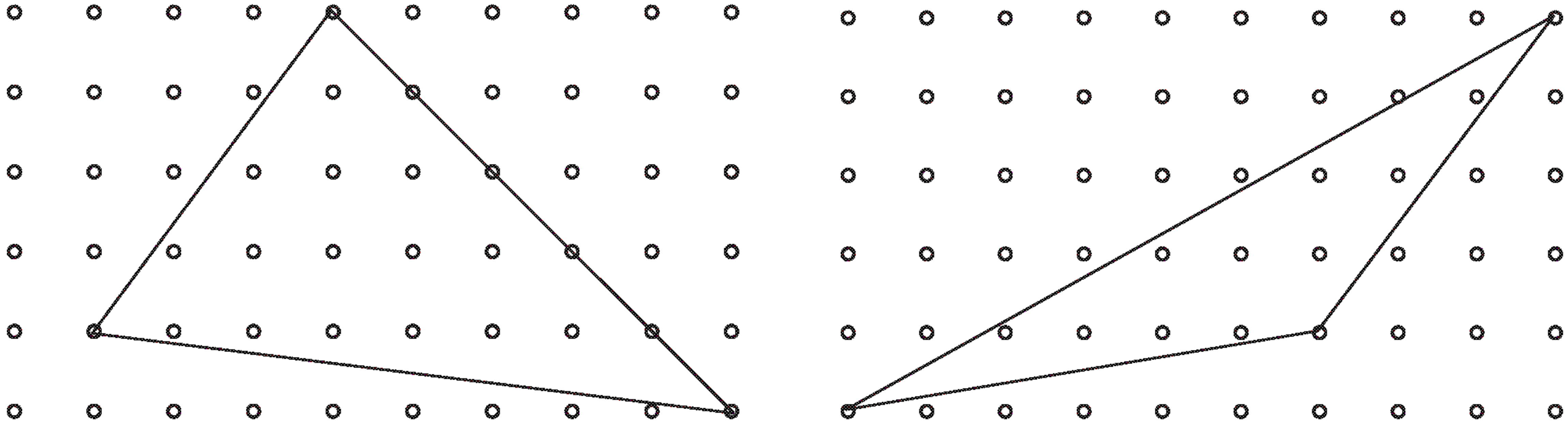
$$a = a_1 + a_2 + d$$
$$b = b_1 + b_2 - 2d - 2$$

These give

$$a + \frac{1}{2}b - 1 = (a_1 + \frac{1}{2}b_1 - 1) + (a_2 + \frac{1}{2}b_2 - 1)$$

and r.h.s. = sum of areas = total area.

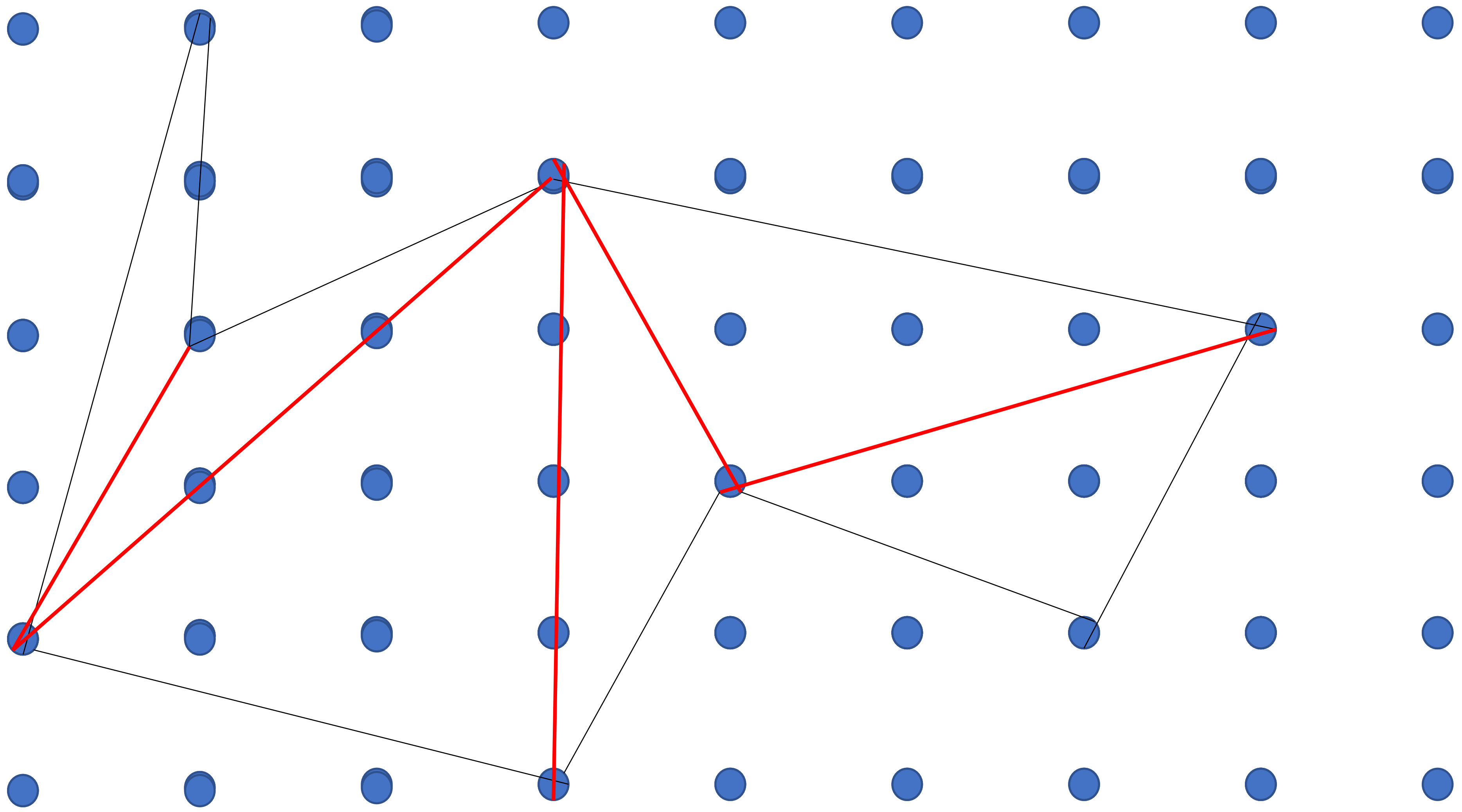
Similarly if the formula works for the whole shape and for one of the parts then it works for the other part.



In each case the triangle can be made from a rectangle by subtracting right-angled triangles and smaller rectangles, for which the formula is known to work.

Every area surrounded by a polygon can be made up of triangles so in fact the formula is always valid for any lattice polygon:

Area = number of lattice points inside + half number of lattice points on the edges minus 1



There are many 'serious' practical applications of Pick's Theorem—chiefly to estimating areas of complicated shapes by placing them on a fine lattice and counting points, usually automatically.

Here is a small application to magic squares.

a	b	c
d	e	f
g	h	k

Let's look at 3 x 3 magic squares. I only require that **all rows, all columns and the two main diagonals add to the same sum, called *s*, and the entries are all positive whole numbers.**

Can you show that $e = \frac{1}{3}s$?

I'll write $s = 3e$.

4	5	c
d	e	f
g	h	k

given s = 15 (so $e = s/3 = 5$) fill in the rest of the magic square.

a	b	c
d	e	f
g	h	k

Given a, b, e (hence the magic sum $3e$), fill in the rest of the magic square in terms of a,b,e.

a	b	$3e - a - b$
$4e - 2a - b$	e	$2a + b - 2e$
$a + b - e$	$2e - b$	$2e - a$

Now, **given only e**, how many choices are there for positive whole numbers *a*, *b* which make **all the entries positive?**

We can answer this question using Pick's Theorem !

We need

$e < a + b < 3e$

$2e < 2a + b < 4e$

$a < 2e$

$b < 2e$

to make all the entries > 0.

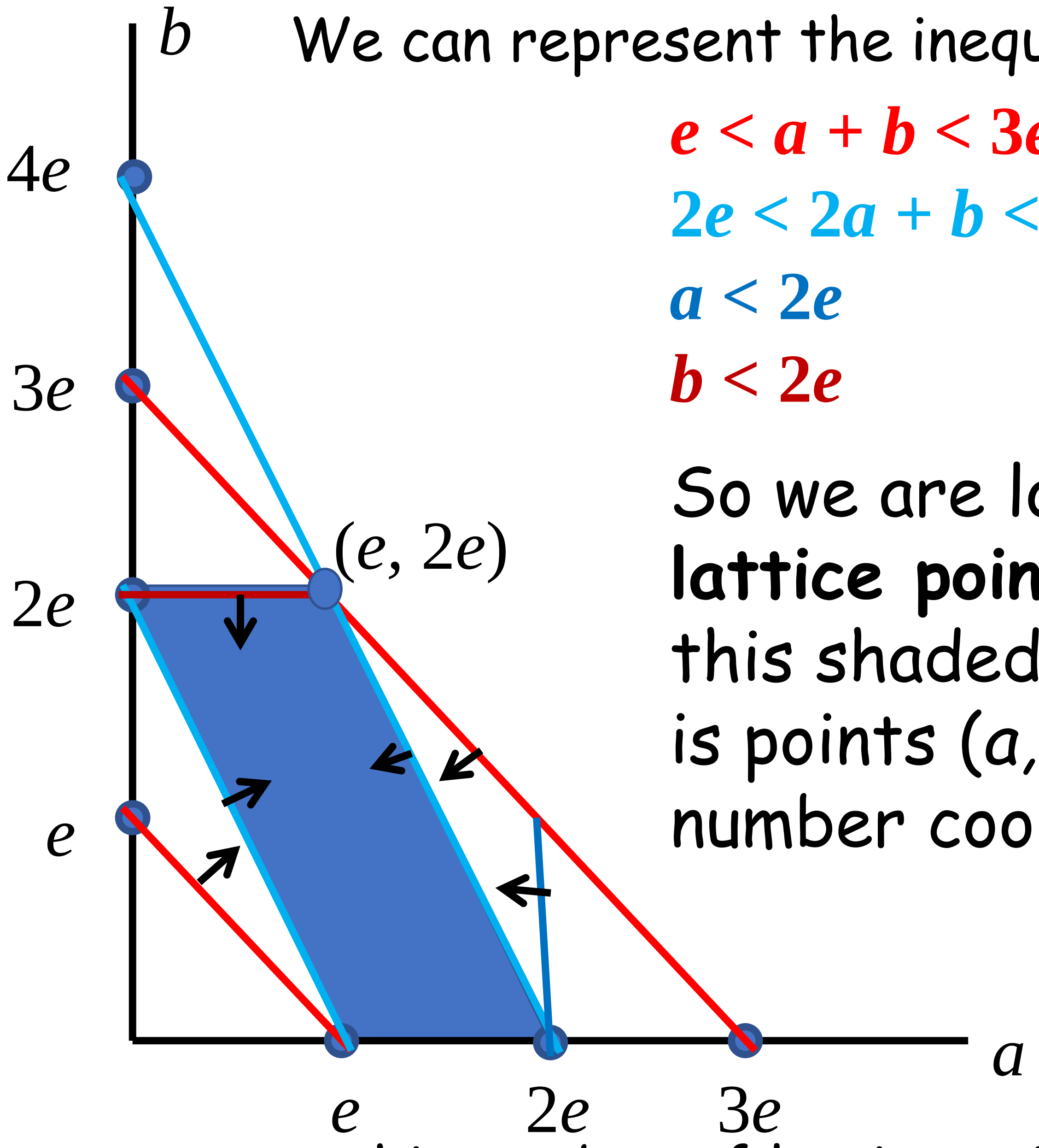
We can represent the inequalities on a diagram:

$e < a + b < 3e$

$2e < 2a + b < 4e$

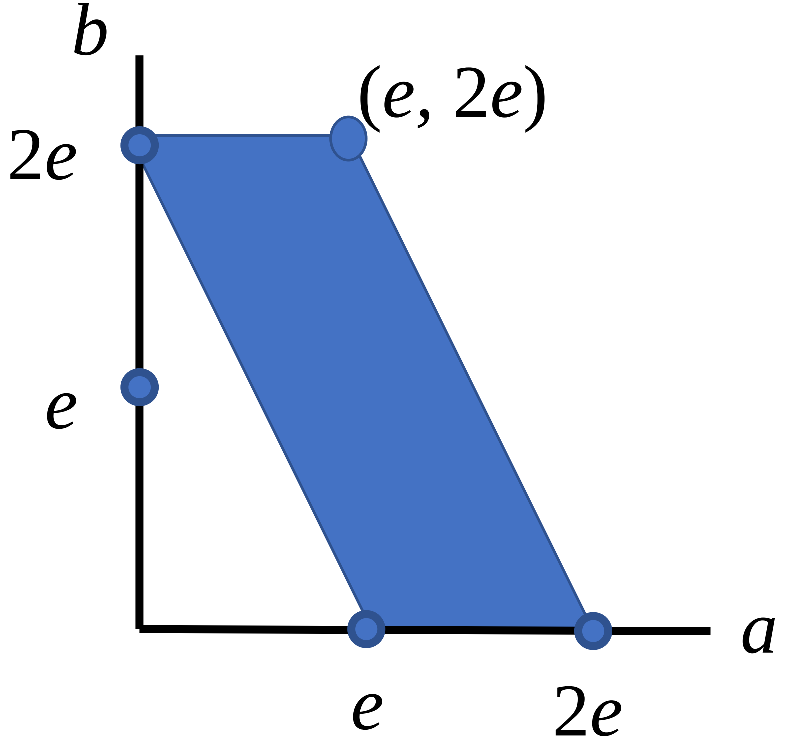
$a < 2e$

$b < 2e$



So we are looking for all the **lattice points** inside (not on) this shaded parallelogram, that is points (*a*,*b*) with whole number coordinates.

How can we count this number of lattice points, that is the number of solutions of the 3x3 magic square when we are given just *e*?



Area = e x 2e = $2e^2$

Number of lattice points on the edges is
$2(e + 1) + 2(e - 1) = 4e$

So number of lattice points inside is given by

Inside + $\frac{1}{2}(4e) - 1 = 2e^2$

so Number of magic squares = $2e^2 - 2e + 1$

e.g. $e = 2$ gives the number of magic squares as 5.
Can you find them all? They are a bit boring!

It is much harder to determine the magic squares with all entries different!

a	b	
	2	

magic sum $3e = 3 \times 2 = 6$

There should be $2e^2 - 2e + 1 = 5$ solutions. Can you find them?

All entries integers > 0, sum of every row, column and the two main diagonals = $3e = 6$

What about $e = 3$? The magic sum is 9 and there are 13 solutions altogether. I don't think there are any solutions with $e = 3$, nor with $e = 4$ which have all entries different.

With $e = 5$, I believe there is exactly one solution with all entries different, up to the obvious symmetries. Can you find it? Are the entries exactly the numbers 1,2,...,9 in that case?

With $e = 6$, I believe that, of the 61 solutions, 3 distinct solutions (allowing for symmetries) have all entries different and one of these has entries which are nine consecutive whole numbers.