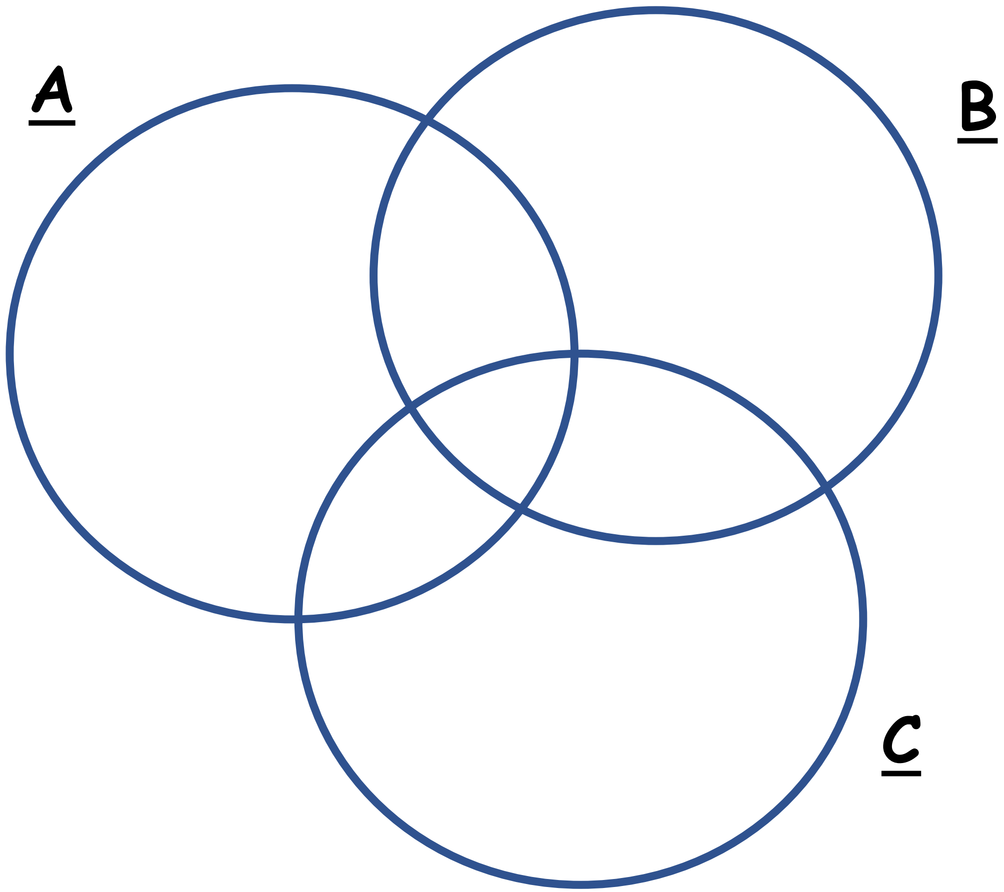
Three sets **A**, **B**, **C**

Is this possible?



A has 30 members,
B has 25 members,
C has 15 members.
The overlap of **A** and **B** has 10
The overlap of **B** and **C** has 12
The overlap of **A** and **C** has 14

What about the last number being 15?

Counting functions

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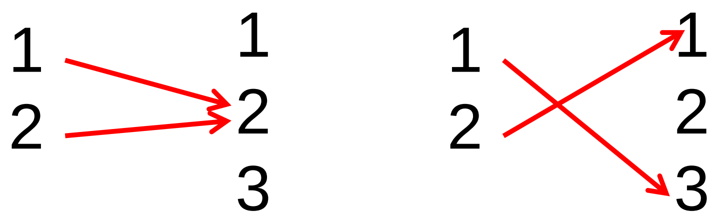
Consider "functions" between two sets.

Define **k** to be the numbers 1,2,3,…,*k*, e.g. **2** means 1,2
And **n** to be the numbers 1,2,3,…,*n*

Example $k = 2$, $n = 3$

These are functions

f: **2** $\to$ **3**



A "function" from **k** to **n** means a rule f which takes each number in the **source** **k** to a definite number in the **target** **n**.

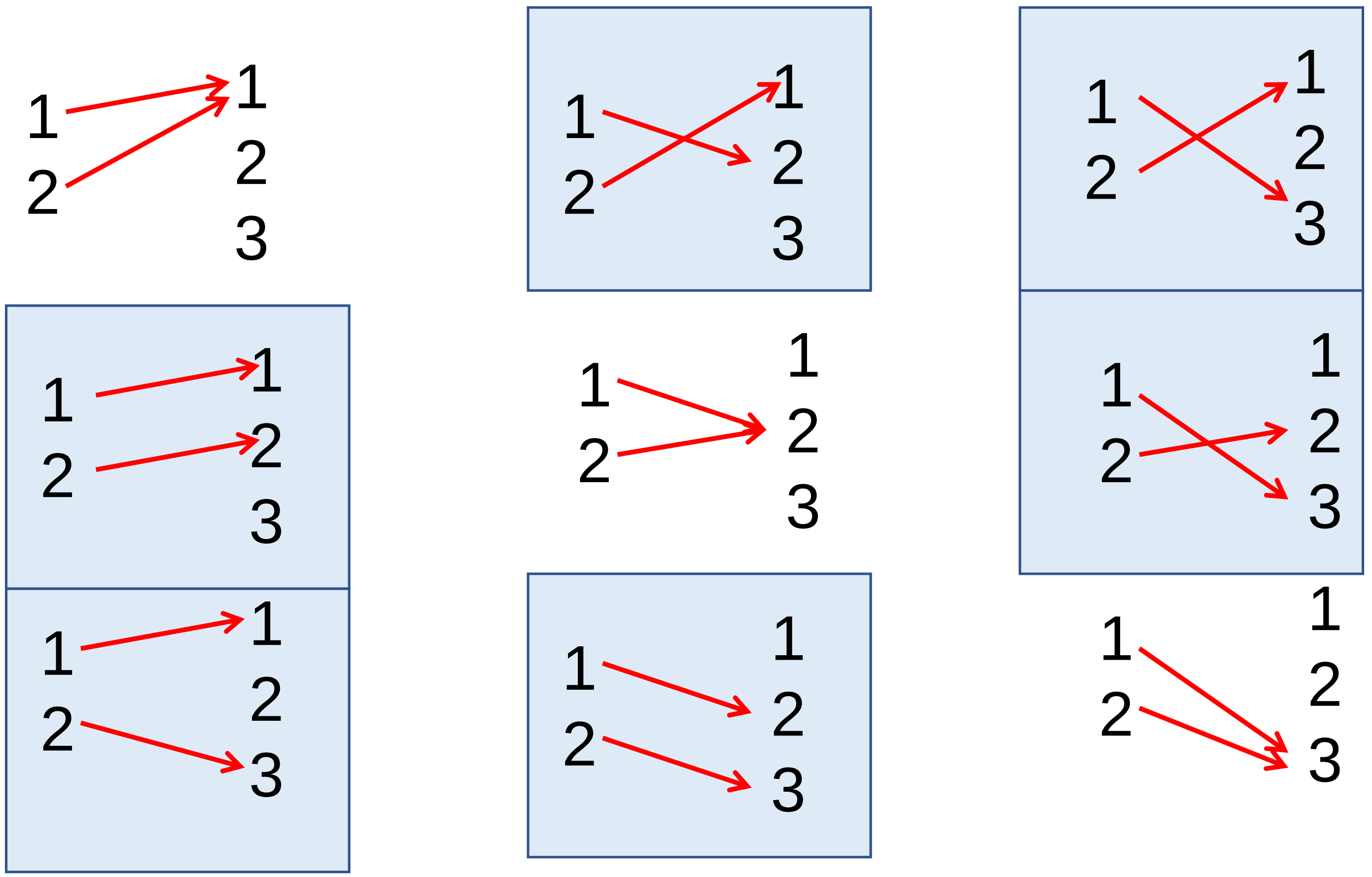
So there is one arrow from each number (**input**) on the left, hitting some number (**output**) on the right

Please draw diagrams of all the functions f: **2** $\to$ **3**

(How many are there?)

There are three choices for where 1 goes and three for where 2 goes: 3x3=9 altogether

maps *f*: **2** → **3**



3 x 3 =9 altogether

3 x 2 = 6 are **injective**

"**injective**": the arrows all go to different outputs.

So the probability of a function being injective is

$$\frac{6}{9} = \frac{2}{3}$$

How many **injective functions** are there between **k** and **n** ?

Maybe you can see that we need $k \leq n$ for this to work.

There are n places where 1 can go,
n-1 places where 2 can go [can't go to same place as 1]
n-2 places where 3 can go [can't go to same place as 1,2]
.... and finally
n-k+1 places where k can go
So $n(n-1)(n-2)....(n-k+1)$ possibilities altogether and the total number of functions is $nxnxnxn$ k *times* = n^k . So the probability of being injective is

$$\frac{n(n-1)(n-2)....(n-k+1)}{n^k}$$

E.g. $k = 2$, $n = 4$ gives 12 injective functions $\underline{\mathbf{2}} \rightarrow \underline{\mathbf{4}}$
(all except 4 of the 16 maps $\underline{\mathbf{2}} \rightarrow \underline{\mathbf{4}}$)

What about functions **23**$\longrightarrow$**365** ?

The probability of all arrows going to different places is, with 23 factors top and bottom

$$\frac{365 \times 364 \times \ldots \times 343}{365 \times 365 \times \ldots \times 365} = 0.4927$$

A shade less than 0.5.

- Inputs: 23 people
- Outputs: birthday (ignoring 29 Feb)
- Arrow: person to birthday

Then just less than an even chance of all birthdays being different.

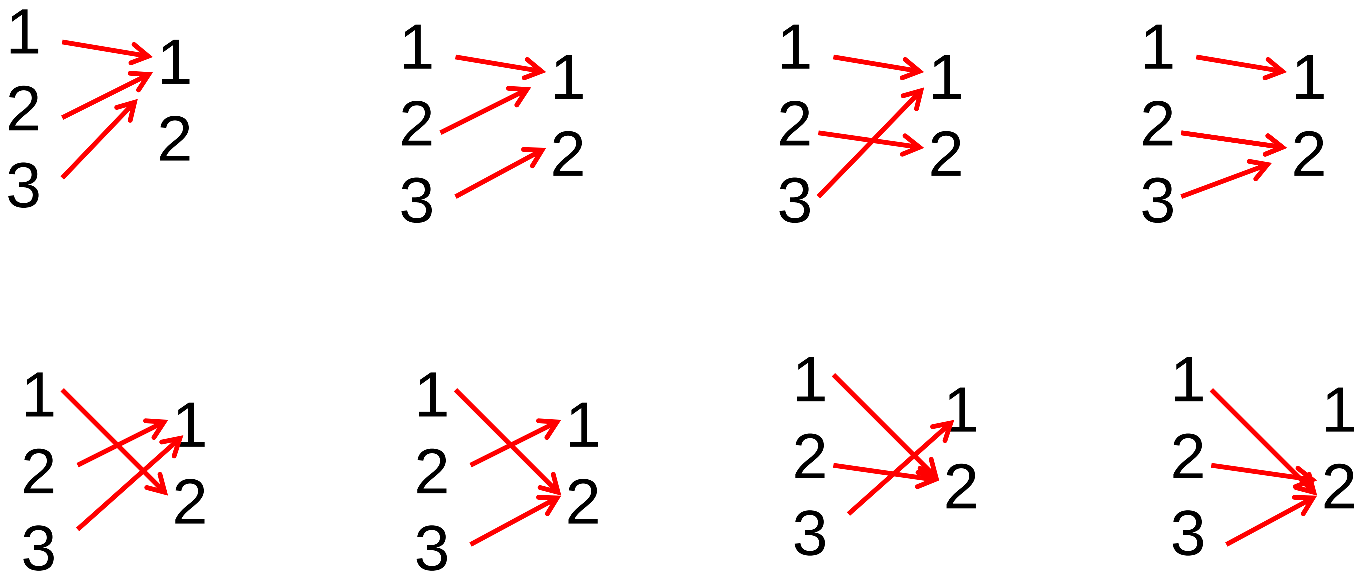
so with 23 people better than even chance that two have the same birthday!

Now for something a little different. These functions have applications in computer science especially coding and decoding.

A function is called **surjective** if every number in the target is hit by at least one arrow. Let's try to count them.

For $f : \underline{\mathbf{k}} \to \underline{\mathbf{n}}$ this requires $k \geq n$

How many surjective functions are there from $\underline{\mathbf{3}}$ to $\underline{\mathbf{2}}$?



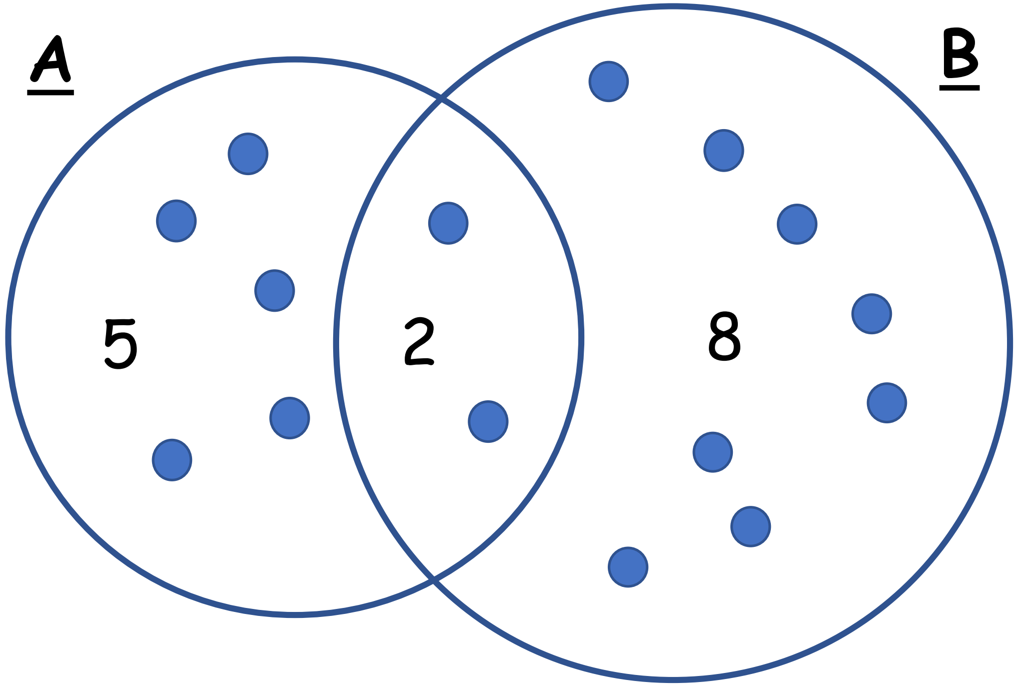
These are all the 2^3 = 8 maps from $\underline{\mathbf{3}}$ to $\underline{\mathbf{2}}$

Maybe you can spot the answer for functions $\underline{\mathbf{4}} \to \underline{\mathbf{2}}$ or indeed for functions $\underline{\mathbf{k}} \to \underline{\mathbf{2}}$: how many are surjective?

Next, we'll try to count surjective functions $\underline{\mathbf{3}} \to \underline{\mathbf{3}}$

Suppose we want to count the total number of things in two sets **A**, **B**

I'll write A for the number of things in **A** etc.



$A = 7$, $B = 10$,

A + B counts everything in the overlap twice.

$15 = 7 + 10 - 2.$

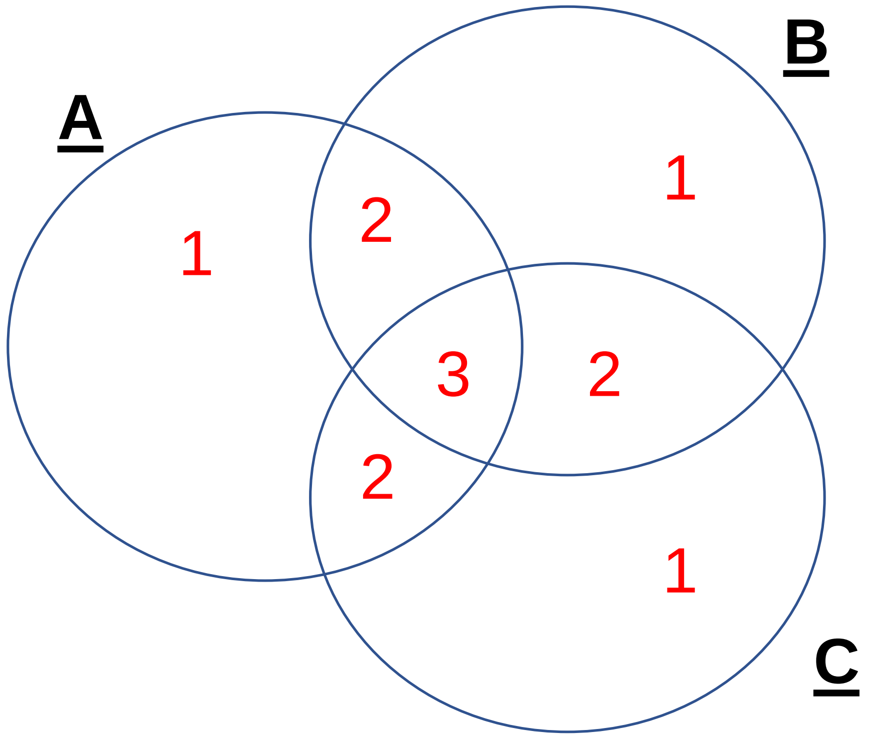
We write **A** ∩ **B** for the overlap: $A \cap B = 2$.

We write **A** ∪ **B** for everything in **A** or **B** or both: $A \cup B = 15$.

$$A \cup B = A + B - A \cap B$$

correction term

Three overlapping sets. If we just add up A + B + C how do we correct this to get the number A U B U C of things in at least one of the three sets?

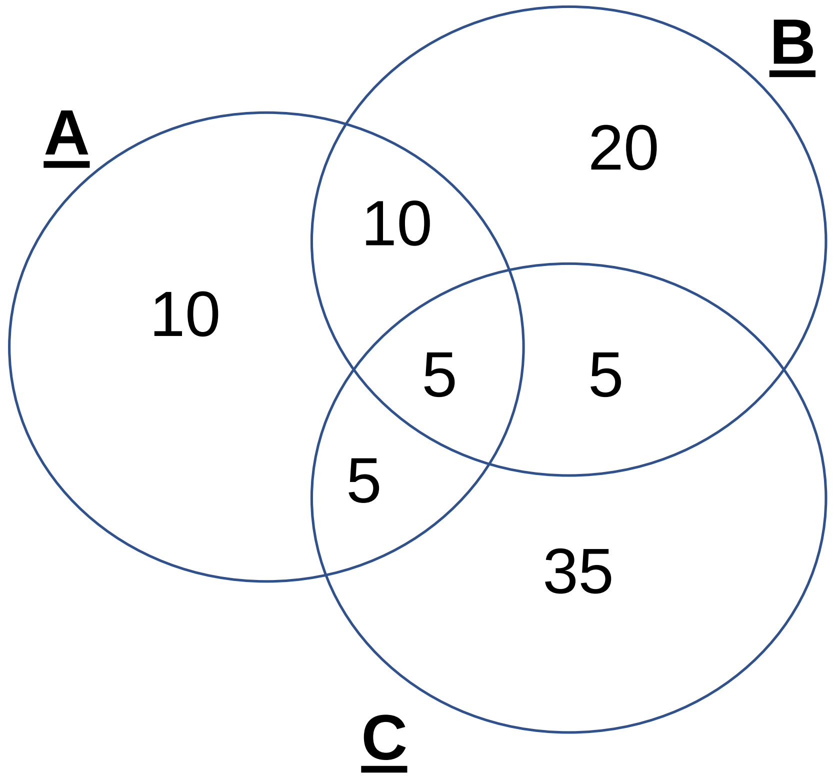


How many times is something counted by just adding up A + B + C ?

Total number of members in all three sets is

$$A \cup B \cup C = A + B + C - (A \cap B) - (B \cap C) - (A \cap C) + (A \cap B \cap C)$$

Example:



These are the actual numbers of people in the various regions

$$A \cup B \cup C = A + B + C - (A \cap B) - (B \cap C) - (A \cap C) + (A \cap B \cap C)$$

$$30 + 40 + 50 - 15 - 10 - 10 + 5$$

$$= 90$$

Problems

$$A \cup B \cup C = A + B + C - (A \cap B) - (B \cap C) - (A \cap C) + (A \cap B \cap C)$$

$$30 + 25 + 20 - 10 - 12 - 14 + x$$

What possible values could x have?

$$A \cup B \cup C = A + B + C - (A \cap B) - (B \cap C) - (A \cap C) + (A \cap B \cap C)$$

$$30 + 25 + 16 - 10 - 12 - 14 + x$$

What possible values could x have?

$$A \cup B \cup C = A + B + C - (A \cap B) - (B \cap C) - (A \cap C) + (A \cap B \cap C)$$

Is this possible?

$$30 + 25 + 15 - 10 - 12 - 14 + x$$

Now we can count the surjective functions $\underline{\mathbf{3}} \rightarrow \underline{\mathbf{3}}$.
There are 27 functions altogether.

1 1
2 2
3 3
with arrows

Here's how:

Let $\underline{\mathbf{A}}$ be all the functions which do **not** hit the number 1
Let $\underline{\mathbf{B}}$ be all the functions which do **not** hit the number 2
Let $\underline{\mathbf{C}}$ be all the functions which do **not** hit the number 3

How many functions in $\underline{\mathbf{A}}$? in $\underline{\mathbf{B}}$? in $\underline{\mathbf{C}}$? $2^3=8$ in each
How many functions in $\underline{\mathbf{A}} \cap \underline{\mathbf{B}}$? (hit neither 1 nor 2) $1^3 = 1$, also $B \cap C = 1$ etc.
How many functions in $\underline{\mathbf{A}} \cap \mathbf{B} \cap \mathbf{C}$? (don't hit anything!) 0
$\underline{\mathbf{A}} \cup \underline{\mathbf{B}} \cup \underline{\mathbf{C}}$ is the functions which fail to hit at least ones of 1,2,3:
These are the *<u>not surjective ones</u>*
$A \cup B \cup C = A + B + C - (A \cap B) - (B \cap C) - (A \cap C) + (A \cap B \cap C)$

$8 + 8 + 8 - 1 - 1 - 1 + 0 = 21$ **not surjective**

so **27 – 21 = 6 surjective**

We can apply the same idea to, say, functions **<u>4</u>** to **<u>3</u>**

Let **A** be all the functions which do **not** hit the number 1
Let **B** be all finctions which do **not** hit the number 2
Let **C** be all the functions which do **not** hit the number 3

How many functions in **A?** in **B**? in **C**?
How many functions in **A∩B**?
How many functions in **A∩B∩C ?**
How many functions in **AUBUC?** [These are the *not surjective functions*]
$A \cup B \cup C = A + B + C - (A \cap B) - (B \cap C) - (A \cap C) + (A \cap B \cap C)$
$2^4 + 2^4 + 2^4 - 1 - 1 - 1 + 0 = 45$
out of the total of $3^4 = 81$ functions are **not surjective**
so 81 – 45 = 36 functions are surjective.

The same idea works for **<u>n</u>** to **<u>3</u>**

General result, using the general inclusion-exclusion principle:number of surjective functions **n** to **k** is (when $n \geq k$)

$$k^n - \binom{k}{1}(k-1)^n + \binom{k}{2}(k-2)^n + \ldots$$

where the sum in continued for k terms (the next one is zero)

If $n = k$ this must equal the number of injective functions (can you see why this is??), namely $k!$, which is a bit surprising....

e.g. $n=k=4$ gives

$4^4 - 4(3^4) + 6(2^4) - 4(1^4) = 256 - 324 + 96 - 4 = 24 = 4!$

Thank You for Your Attention!