1 4 6 9 12 14 17 19 22

2 7 10 15 20

3 4 11 12 16 17

5 6 7 18 19 20

8 9 10 11 12

13 14 15 16 17 18 19 20

21 22

Choose a number between 1 and 20

Add up the first (leftmost) number in each row containing your number. What answer do you get?

What are the first numbers in each row? Do you recognize them? What's going on?

Where would 21 and 22 fit in the table?

Fibonacci representations and a game with two piles of counters

University of Liverpool Maths Club
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Peter Giblin
pjgiblin@liv.ac.uk

Fibonacci numbers: 1, 2, 3, 5, 8, 13, 21, 34, 55,89, 144,.....

NOTE that I do NOT start the sequence with 1,1,2,3,5 but with 1,2,3,5...

The *Fibonacci representation* of any whole number N:

Subtract from N the largest Fibonacci number ≤ N.

From that subtract the largest Fibonacci number etc.

E.g. 130 - 89 = 41

41 - 34 = 7

7-5 = 2

2-2 = 0

Fibonacci representation	1	0	1	0	0	0	1	0	1	0		
	89	+	34		+		5	+	2		=	130

We don't use two consecutive Fibonacci numbers here.

Try some examples: does the same happen for you?

Write u_n for the *n*th Fibonacci numbers, $u_1 = 1$, $u_2 = 2$ etc.
Thus $u_{n+1} = u_n + u_{n-1}$ ($n > 1$) is the rule for forming the numbers.
Suppose $u_n \leq N < u_{n+1}$ that is N lies between the nth and (n+1)st Fibonacci numbers, so u_n is the largest Fibonacci number $\leq N$.
Then $0 \leq N - u_n < u_{n+1} - u_n = u_{n-1}$.

So taking u_n away from N gives an answer < the previous Fibonacci number u_{n-1} and we won't use u_{n-1} at the next subtraction.

So consecutive u_n are never used in the **Fibonacci representation** of N.

1 2 3 5 8 13 21 34 55 89 144 233 377 610 987

Find the Fibonacci representation (FR) of

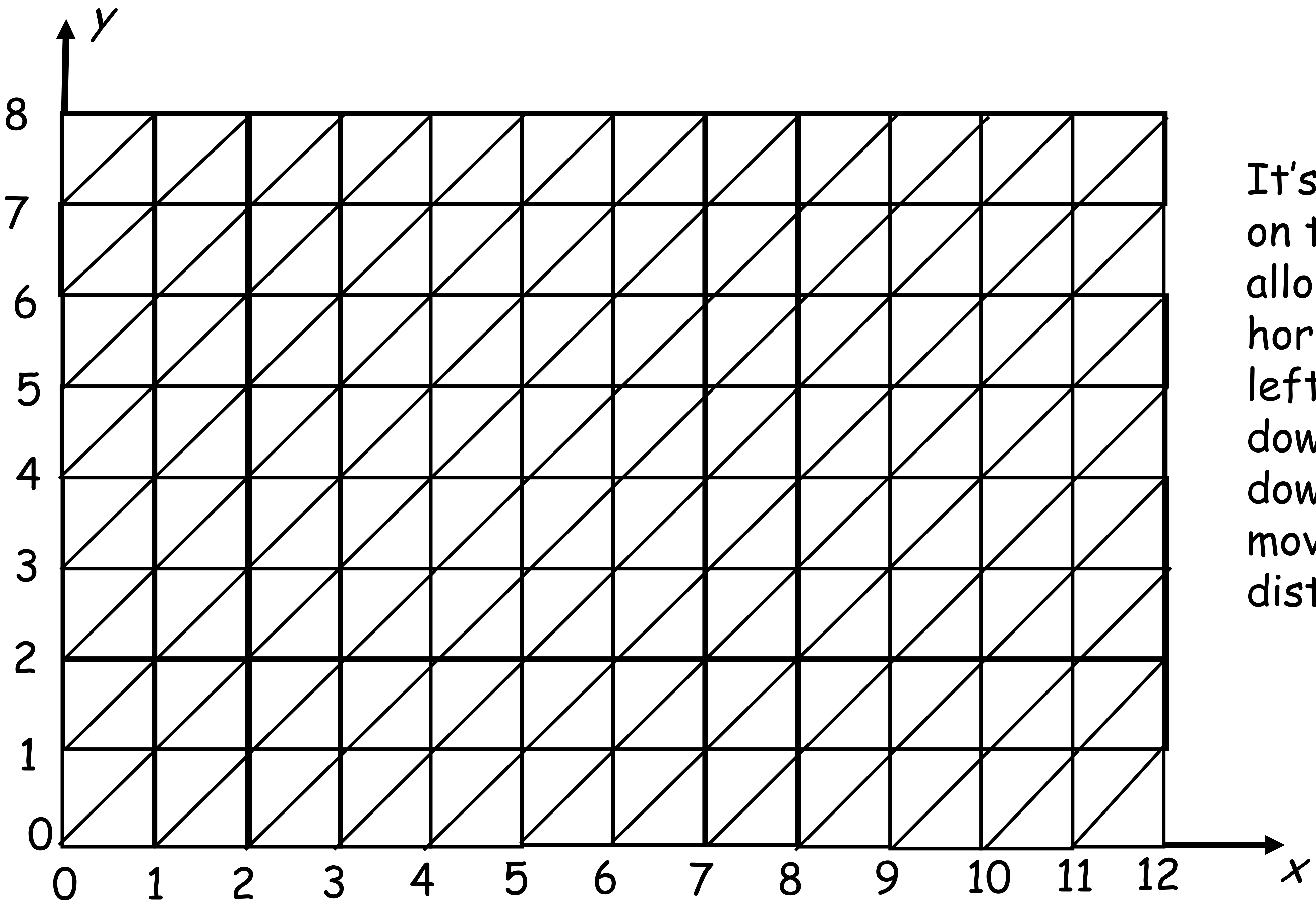
500	377 + 89 + 34	FR	1001010000000
750	610 +89 + 34 + 13 + 3 + 1	FR	10001010100101
1000	987 + 13	FR	1000000000100000

But now I want to apply the Fibonacci representation to a **game** called **tzyan-she-tzy**.

Start with two piles of counters, say x in one pile and y in the other pile.

Two people move alternately. At each move players **either** take any number (> 0) of counters from **one** pile or **the same number** (> 0) from **both** piles.

The last person to move is the winner.



It's helpful to play on this grid, where allowed moves are horizontally to the left, or vertically down, or diagonally down, in all cases moving through any distance.

At each move **either** take any number of counters from **one** pile or **the same number** from **both** piles.

Can you find some **winning positions** in this game?

A **winning position for me** is given by two piles x and y of counters such that, if it is my opponent's move, then **no matter what the he/she does**, I can win, provided I continue to play correctly.

For example, (x,y)=(2,1) is such a winning position for me. Clearly if (x,y) is a winning position for me so is (y,x).
Then there cannot be any other winning positions with $x = 1$ or 2, nor with $y = 1$ or 2.

- After my move it must be impossible for my opponent to make a move to give a winning position for him/her.

- Supposing the counters are not in a winning position for me and it is my move, I must be able to put the counters in a winning position.

1 2 3 5 8 13 21 34 55 89 144.......

Consider pairs of numbers (x,y) according to the following rule:

y is any number with FR ending (on the right) in an even number of 0s (possibly zero 0s);
x is given by placing one 0 at the right hand end of the FR of y

For example

$y = 4 = 3+1$, FR = 101 (ending in zero 0s)
x has FR 1010 so $x = 5 + 2 = 7$ $(x,y) = (7,4)$

$y = 11 = 8 + 3$, FR = 10100 (ending in two 0s)
x has FR 101000 so $x = 13 + 5 = 18$, $(x,y) = (18, 11)$.

(If y is an odd-numbered Fibonacci number then x is the next Fibonacci number, e.g. $y = 8$ gives $x = 13$.)

Pairs (x,y) with $x > y$, formed by the above rule:

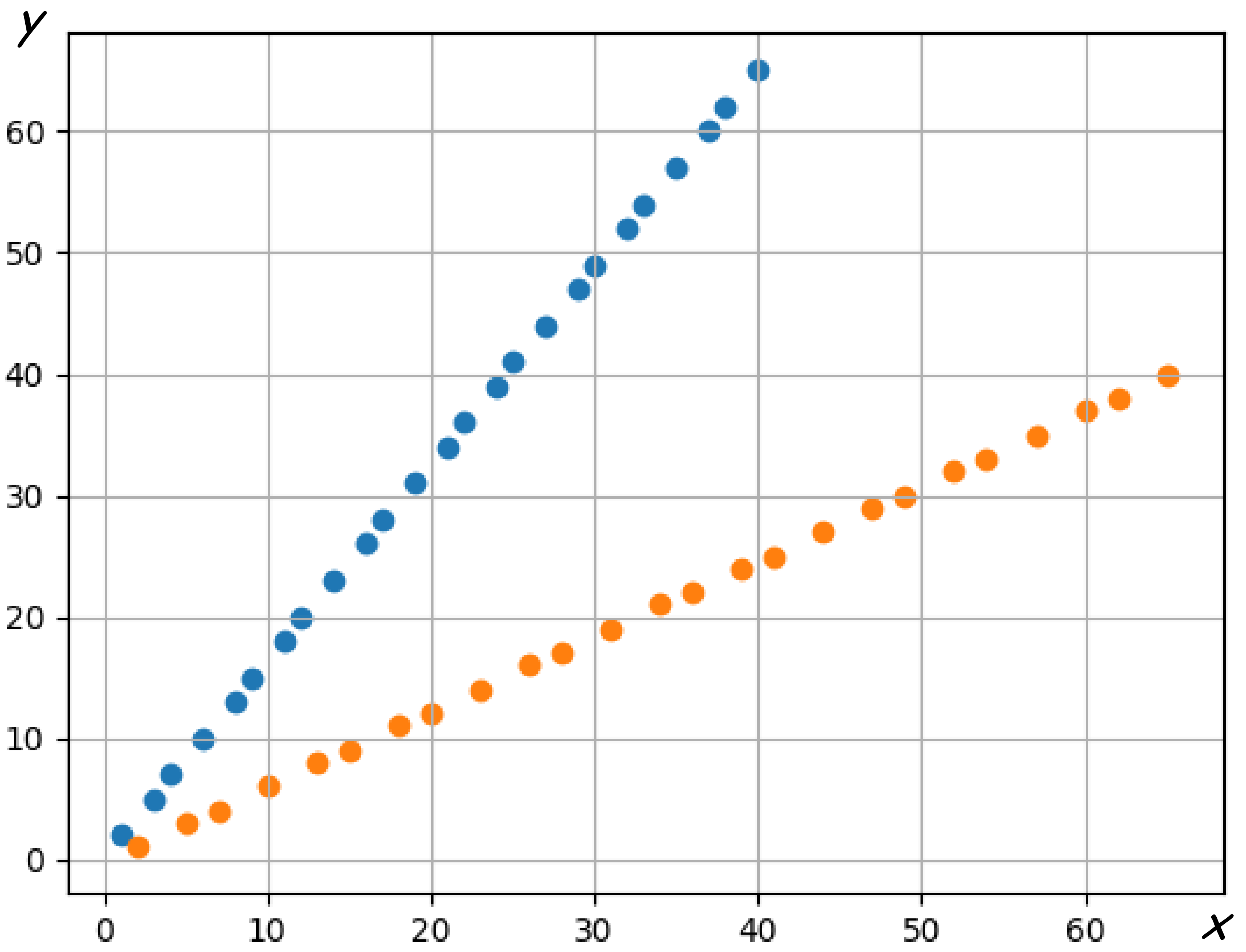
2	1
5	3
7	4
10	6
13	8
15	9
18	11
20	12
23	14
26	16
28	17
31	19
34	21
36	22
39	24
41	25
44	27
47	29
49	30
52	32
54	33
57	35
60	37
62	38
65	40

Together with corresponding (y,x) it can be proved that they are exactly the winning positions!!

Every number occurs just once in this table, and of course just once in the corresponding (y,x) table.

Amazing!

Winning positions almost lie on two straight lines through the origin



In fact the winning positions are exactly the pairs

$$(x, y) = \left(\left\lfloor n\frac{1+\sqrt{5}}{2} \right\rfloor, \left\lfloor n\frac{3+\sqrt{5}}{2} \right\rfloor \right),$$

together with their reflexions (y, x) in the line $x = y$. Here $\lfloor a \rfloor$ means the greatest integer $\leq a$. These points almost lie on the lines

$$y = \frac{\sqrt{5}+1}{2}x, \quad y = \frac{\sqrt{5}-1}{2}x$$

These gradients are the 'golden ratio' and its reciprocal.

There is a full discussion of this, with proofs, in the book **Fibonacci Numbers** by Nicolai N Vorobiev, translated from the Russian by Mircea Martin (Birkhauser 2002, ISBN 978-3-0348-8107-4).

There is a part of this book available online which covers the end of the discussion of tzyan-she-tzy.

But not alas the whole story!

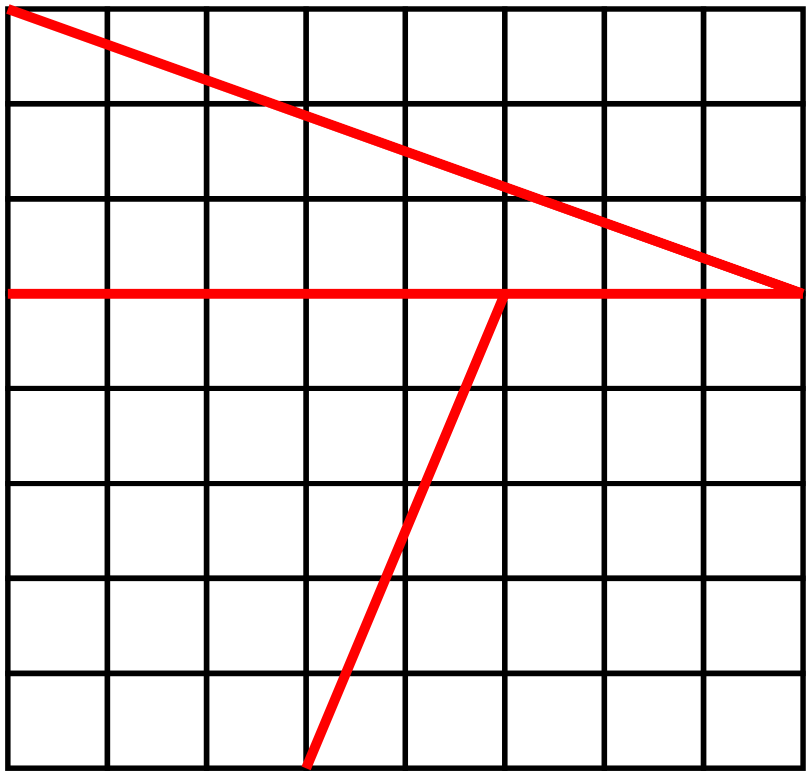
Thank you for your attention!

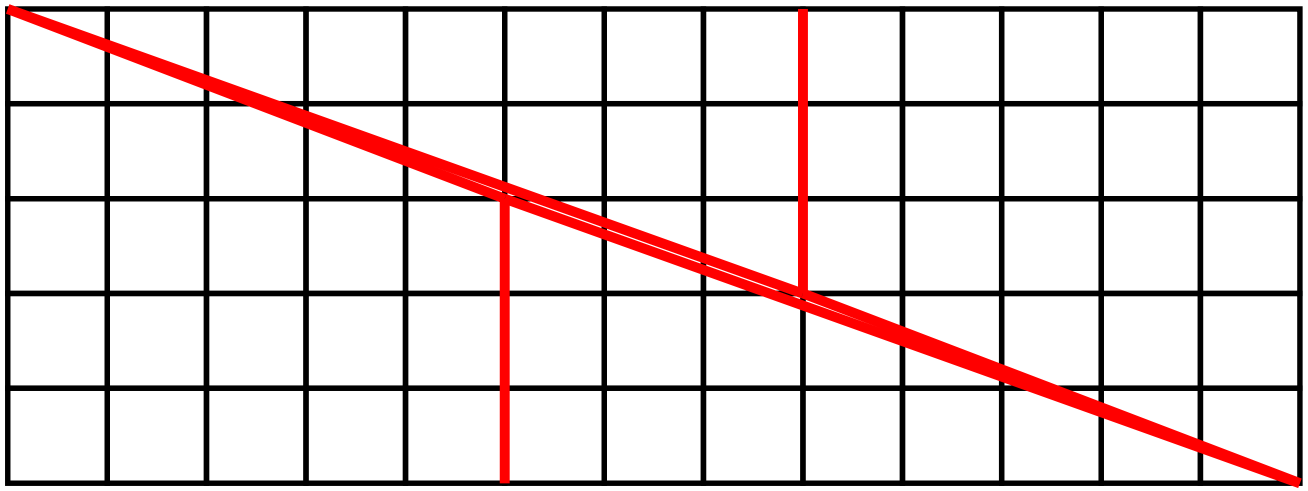
Extra Note 1

1 2 3 5 8 13 21 35 56 ....

$2^2 = 1 \times 3 + 1,\qquad 3^2 = 2 \times 5 - 1,\qquad 5^2 = 3 \times 8 + 1,\qquad 8^2 = 5 \times 13 - 1$

and so on; the pattern always continues (can you prove this ?!). Consider the fourth example above.





Cut and re-assemble. You can almost be convinced that $8^2 = 5 \times 13$. With

$$13 \times 35 = 21^2 + 1$$

it's even harder to see the little gap!

Extra Note 2

Note that there are usually many other ways to express a number as a sum of distinct Fibonacci numbers, e.g. 130=89+21+13+5+2.

But the *Fibonacci representation* (FR) is obtained as above. It never uses consecutive Fibonacci numbers.

There are some numbers for which the FR is the *only* expression as a sum of Fibonacci numbers 1,2,3,5,8,13,... (remember I don't start the sequence with 1,1,2,3,5,...)

e.g. 12 = 8 + 3 + 1 is the only way.