Playing with positive polynomials

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What is a polynomial?

Some examples of polynomials:

$3x - 2, \quad x^2 + 1, \quad 2x^3 - 5x^2 + x - 1, \quad ax^2 + bx + c,$

$$a_5 x^5 + a_4 x^4 + a_3 x^3 + a_2 x^2 + a_1 x + a_0$$

The following are not polynomials: $\frac{1}{x} \quad \sqrt{x} \quad \sin x$.

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The highest power of x in a polynomial is called its **degree**.
The term of highest degree is called its **leading term**.
Polynomials of degree 1 are said to be **linear** (Think why!) Those of degree 2 are called **quadratics**, those of degree 3 are called **cubics**, . . .

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The numbers multiplying the powers of x (e.g. $a_5, a_4, \ldots, a_0$) are called **coefficients**. Note that coefficients can be zero, e.g. the coefficient of x in $x^2 + 1$ is zero.
Sometimes we shall write $f(x)$ or $g(x)$ to denote a polynomial.
E.g. $f(x) = ax^2 + bx + c$.

Polynomials are pleasant

1. They are a bit like whole numbers. You can add and subtract them (and get the same sort of creature). You can also multiply them. You may or may not be able to divide them. ($7 \div 3$ is not a whole number. $\frac{x^2+1}{x+1}$ is not a polynomial, but $\frac{x^2-1}{x+1} = x - 1$ is a polynomial. This means that factorization of ploynomials is interesting.

2. We can picture polynomials by drawing graphs.

How to multiply polynomials

Think what the product looks like and what its leading term is.
$(x^2+3x+4)(2x^2+x+5) = 2x^4 + \ldots$.
Think where the x^3 term in the answer comes from:

$$\begin{aligned}(x^2+3x+4)(2x^2+x+5) &= 2x^4 + (x^2.x + 3x.2x^2) + \ldots \\ &= 2x^4 + 7x^3 + \ldots\end{aligned}$$

Next, think where the x^2 term in the answer comes from and so on:

$$\begin{aligned}(x^2+3x+4)(2x^2+x+5) &= 2x^4 + 7x^3 + (x^2.5 + 3x.x + 4.2x^2) + \ldots \\ &= 2x^4 + 7x^3 + 16x^2 + \ldots \\ &= 2x^4 + 7x^3 + 16x^2 + (3x.5 + 4.x) + \ldots \\ &= 2x^4 + 7x^3 + 16x^2 + 19x + 4.5 \\ &= 2x^4 + 7x^3 + 16x^2 + 19x + 20\end{aligned}$$

With practice, you can write the answer straight down.

Do we need long multiplication?

$(x^2 + 3x + 4)(2x^2 + x + 5) = 2x^4 + 7x^3 + 16x^2 + 19x + 20.$

Putting $x = 10$ and remembering about carrying, gives

$$134 \times 215 = 28,810$$

Think about polynomial multiplication and amaze your friends with your mental arithmetic! Can you write down the answer to:

$$327 \times 452$$

Positive polynomials

What might it mean to say that a polynomial $f(x)$ is positive?
Possible answers:
(1) All the coefficients of $f(x)$ are positive. For example

$$f(x) = x^2 + 3x + 2.$$

(2) $f(x) > 0$ for all $x \geq 0$.

Think of the graph $y = f(x)$ in each case.

Suppose that $f(x)$ and $g(x)$ are polynomials that satisfy (1).
What can you say about (i) $f(x) + g(x)$, (ii) $f(x)g(x)$?

Suppose that $f(x)$ and $g(x)$ are polynomials that satisfy (2).
What can you say about (i) $f(x) + g(x)$, (ii) $f(x)g(x)$?

Notice that (1) $\Longrightarrow$ (2).

Does (2) $\Longrightarrow$ (1)?

If $f(x) > 0$ for all $x \geq 0$, is each coefficient of $f(x)$ positive?

1. Suppose that $f(x) = ax + b$.
If $ax + b > 0$ for all $x \geq 0$, must both a and b be positive?

2. Suppose that $f(x) = a(x + d)(x + e)$.
If $a(x+d)(x+e) > 0$ for all $x \geq 0$, must a, d and e all be positive?

3. Suppose that $f(x) = ax^2 + bx + c$.
If $ax^2 + bx + c > 0$ for all $x \geq 0$, must a, b and c all be positive?

What do you think happens for

(i) cubic polynomials that factorize?

(ii) general cubics?

Problem

Can you find an example of a cubic

$$(ax^2 - bx + c)(x + N)$$

with (i) all its coefficients positive (when multiplied out)
and (ii) N, a, b and c all positive?
Perhaps there is no such example!

First idea: Choose any positive values you like for a, b and c. Then see if you can find a suitable N.

Second idea: Multiply the two brackets and write down what inequalities you get from knowing that all the coefficients are positive. What do these inequalities tell you about N?

Solution

Given that a, b, c are all positive. Suppose there is a suitable N.

$$(ax^2 - bx + c)(x + N) = aNx^3 + (aN - b)x^2 + (c - bN)x + cN.$$

If all the coefficients on the right are positive (or zero), then

$$\begin{aligned} aN &\geq 0 && (1) \\ aN - b &\geq 0 && (2) \\ c - bN &\geq 0 && (3) \\ \text{and} \quad cN &\geq 0 && (4) \end{aligned}$$

From (2) and (3),

$$\frac{b}{a} \leq N \leq \frac{c}{b}$$

So $b^2 \leq ac$. Conversely, if $b^2 \leq ac$, then $\frac{b}{a} \leq \frac{c}{b}$ and we can find a number N that lies between these fractions. Hence equations (1) to (4) are satisfied.

An example

$$(x^2 - x + 3)(x + 2) = x^3 + x^2 + x + 6$$

"Accentuate the positive,
Eliminate the negative"

(Bing Crosby)

Let's call $x + 2$ an **eliminator** of the quadratic $x^2 - x + 3$.

Think about what we have proved

Given a quadratic $ax^2 - bx + c$ in which a, b, c are all positive. Its graph lies above the $x - axis$ if and only if the equation $ax^2 - bx + c = 0$ has no real roots. This happens if and only if $b^2 < 4ac$. We have shown that, when $b^2 \leq ac$, there is a first degree polynomial $x + N$ that, when it multiplies our quadratic, all the negative coefficients are eliminated. Also, when $b^2 > ac$, there is no such first degree polynomial.

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Questions: (1) What happens when $ac \leq b^2 < 4ac$?
(2) When does $f(x) \equiv ax^2 - bx + c$ have a second degree eliminator?
Question (2) is only of interest when our quadratic has no first degree eliminator. For example, If all the coefficients of $(x+2)f(x)$ are positive, then all the coefficients of $(x+3)(x+2)f(x)$ must be positive, so $(x+2)(x+3)$ will be a second degree eliminator of $f(x)$.

A second look at first degree eliminators

As usual $a, b, c > 0$. Suppose that $a_1 x + a_0$ is an eliminator of $ax^2 - bx + c$. Then all the coefficients of $(ax^2 - bx + c)(a_1 x + a_0)$ are positive or zero. So

$$\begin{aligned} aa_1 &\geq 0 && (5) \\ -ba_1 + aa_0 &= p_1 \geq 0 && (6) \\ ca_1 - ba_0 &= p_0 \geq 0 && (7) \\ ca_0 &\geq 0 && (8) \end{aligned}$$

where $p_1, p_2 \geq 0$. From (8) and (5), $a_0, a_1 \geq 0$.

$$\begin{aligned} b(6) + a(7) : (ac - b^2)a_1 &= bp_1 + ap_0 && (9) \\ c(6) + b(7) : (ac - b^2)a_0 &= cp_1 + bp_0 && (10) \end{aligned}$$

So, if a_0 and a_1 exist, then $b^2 \leq ac$.

Second degree eliminators

As usual $a, b, c > 0$. Suppose that $ax^2 - bx + c$ has no first degree eliminator. Then $b^2 > ac$. Suppose also that $a_2x^2 + a_1x + a_0$ is a second degree eliminator. Then all the coefficients of $(ax^2 - bx + c)(a_2x^2 + a_1x + a_0)$ are positive or zero. So

$$\begin{aligned}
aa_2 &\geq 0 && (11)\\
-ba_2 + aa_1 &= p_2 \geq 0 && (12)\\
ca_2 - ba_1 + aa_0 &= p_1 \geq 0 && (13)\\
ca_1 - ba_0 &= p_0 \geq 0 && (14)\\
ca_0 &\geq 0 && (15)
\end{aligned}$$

where $p_2, p_1, p_0 \geq 0$. From (11), (15) and (12), $a_2, a_1, a_0 \geq 0$. Manipulate equations (12) to (14) as follows:

$(b^2 - ac)(12) + ab(13) + a^2(14)$

$bc(12) + b^2(13) + ab(14)$

$c^2(12) + bc(13) + (b^2 - ac)(14)$

Second degree eliminators - continued

As usual $a, b, c > 0$. Also $b^2 > ac$ and $a_0, a_1, a_2 \geq 0$.

$$\begin{aligned} -ba_2 + aa_1 &= p_2 \geq 0 && (12) \\ ca_2 - ba_1 + aa_0 &= p_1 \geq 0 && (13) \\ ca_1 - ba_0 &= p_0 \geq 0 && (14) \end{aligned}$$

$(b^2 - ac)(12) + ab(13) + a^2(14)$
$bc(12) + b^2(13) + ab(14)$:
$c^2(12) + bc(13) + (b^2 - ac)(14)$:

$b(2ac - b^2)a_2 = (b^2 - ac)p_2 + abp_1 + a^2 p_0$
$b(2ac - b^2)a_1 = bcp_2 + b^2 p_1 + abp_0$
$b(2ac - b^2)a_0 = c^2 p_2 + bcp_1 + (b^2 - ac)p_0$
Hence, a necessary and sufficient condition for $ax^2 - bx + c$ to have a second degree eliminator (and no first degree eliminator), is that $ac < b^2 \leq 2ac$.

What comes next?

When $a, b, c > 0$, the graph of $ax^2 - bx + c$ lies above the x-axis if and only if $b^2 < 4ac$. The table gives the condition for $ax^2 - bx + c$ to have an eliminator of degree n.

n	Condition
1	$b^2 \leq ac$
2	$b^2 \leq 2ac$
3	

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1	$b^2 \leq ac$
2	$b^2 \leq 2ac$
3	$b^2 \leq \frac{1}{2}(3 + \sqrt{5})ac$

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n	Condition
1	$b^2 \leq ac$
2	$b^2 \leq 2ac$
3	$b^2 \leq \frac{1}{2}(3+\sqrt{5})ac$
4	$b^2 \leq 3ac$
$\dots$	$\dots\dots\dots\dots$
$\dots$	$\dots\dots\dots\dots$
∞	$b^2 < 4ac$

What is this strange sequence of numbers on the right?

$1, 2, \frac{1}{2}(3+\sqrt{5}), 3, \dots \longrightarrow 4$

The nth term in the sequence is $2 + 2\cos(\frac{360}{n+2})^\circ$. As $n \to \infty$, the nth term tends to 4.