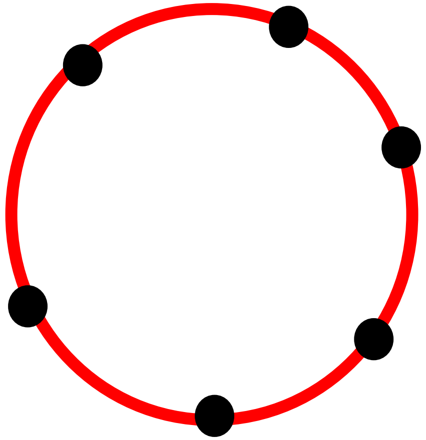
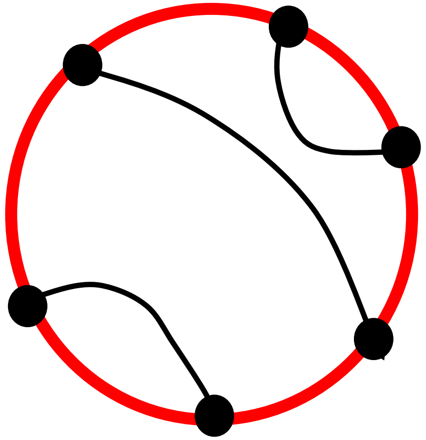
Maths Club Starter

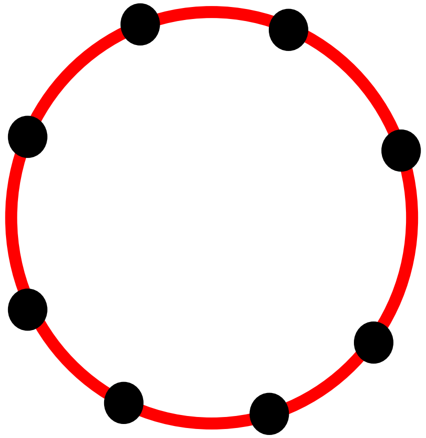
Six points round a circle



Here is one way to join them in pairs, by curves <u>inside the circle</u> <u>which do not cross</u>



There are four other ways. Can you find them?



Then try counting the number of ways to join in pairs (same rules) with eight points

Counting with Catalan
University of Liverpool Maths Club
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If there are any Catalan speakers in the audience I have to confess straight away that I am not talking about

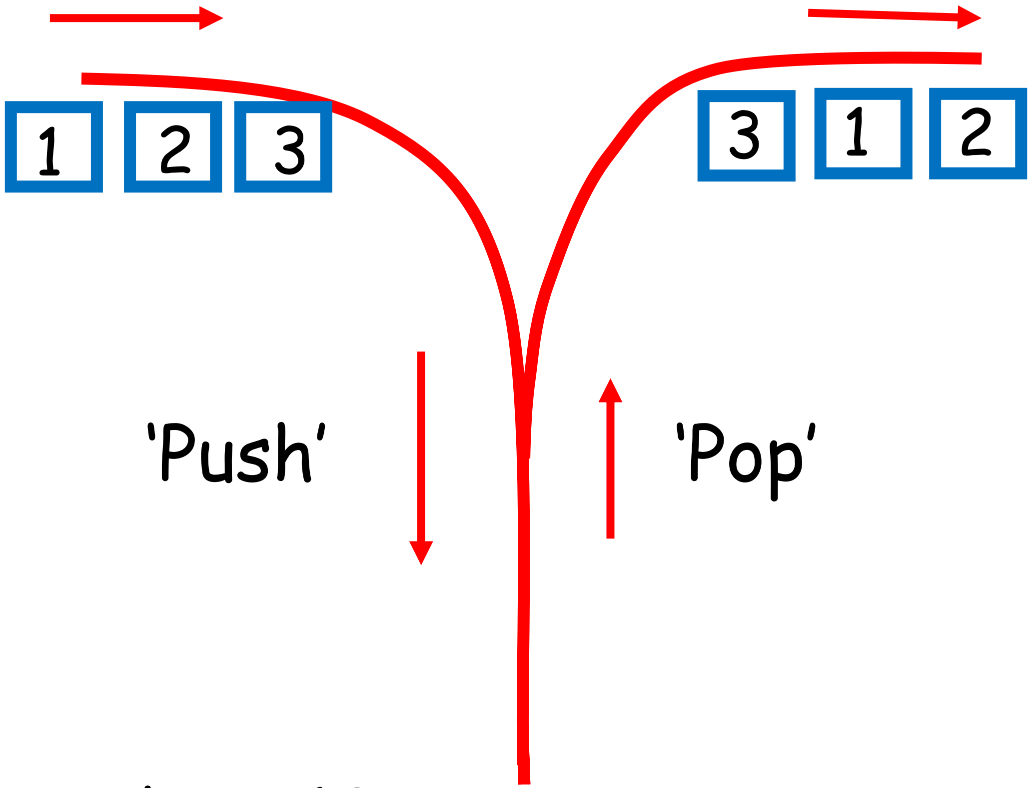
- 1 – un
- 2 – dos
- 3 – tres
- 4 – quatre
- 5 – cinc
- 6 – sis
- 7 – set
- 8 – vuit
- 9 – nou
- 11 – onze
- 12 – dotze
- 13 – tretze
- 10 – deu
- 14 – catorze
- 15 – quinze
- 16 – setze

Instead I am talking about this man

Eugène Catalan
1814-1894

whose numbers appear everywhere in mathematics and computing

Let's begin with model railways.....



Suppose we have three carriages on the left and want to move them to the right.
We could push 3, then push 2, then pop 2.
Then push 1, pop 1 and pop 3.

This gives the order 312
There are 6 possible orderings (permutations) of 1,2,3
Which other orders can be achieved by shunting (pushing and popping)?

The one that cannot be achieved is 2,3,1
So 5 of the 6 can be achieved: there are 5 '**stack-sortable**' permutations of 1,2,3. (There are connections with computer memories.)

For any number of carriages:

...a....b....c.... $\longrightarrow$ ....b....c....a.... is impossible

(In fact this is the **only** obstacle to stack-sorting.)

Which permutations of 1, 2,3,4 have <u>no</u> such triples a,b,c

...a....b....c.... $\longrightarrow$ ....b....c....a.... ?

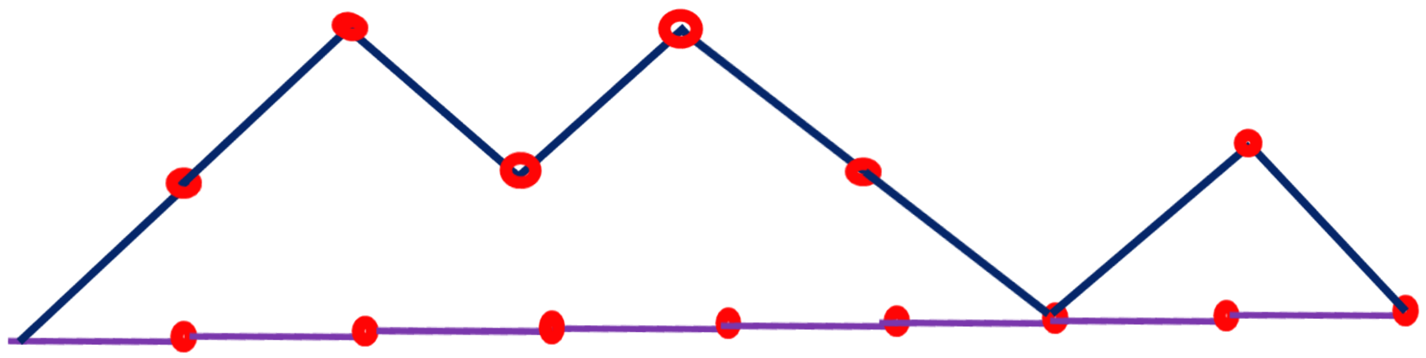
Here are the 24 permutations of 1,2,3,4

1 2 3 4	2 1 3 4	3 1 2 4	4 1 2 3
1 2 4 3	2 1 4 3	3 1 4 2	4 1 3 2
1 3 2 4	2 3 4 1	3 2 1 4	4 2 1 3
1 3 4 2	2 3 1 4	3 2 4 1	4 2 3 1
1 4 2 3	2 4 1 3	3 4 1 2	4 3 1 2
1 4 3 2	2 4 3 1	3 4 2 1	4 3 2 1

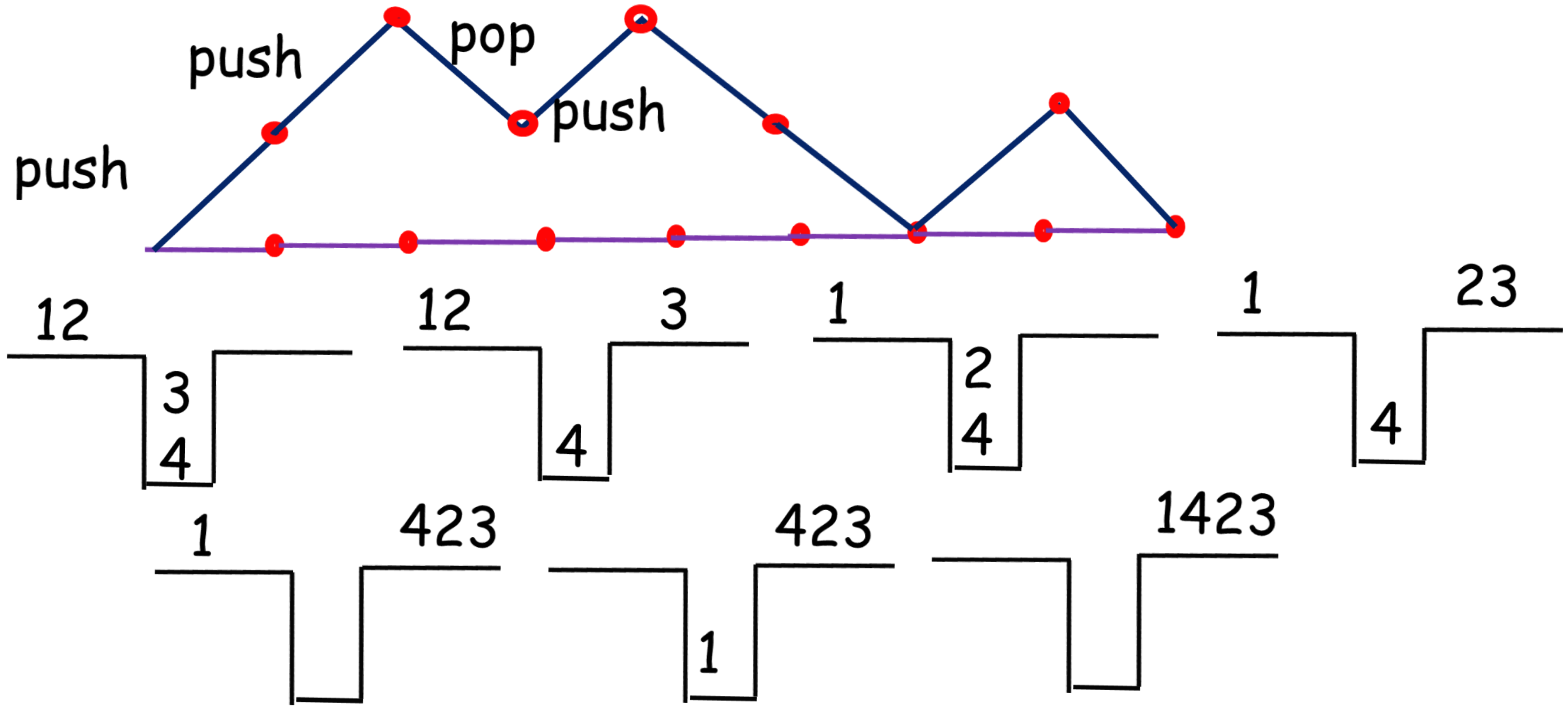
$C_4 = 14$
$C_5 = 42$
$C_6 = 132$

For n carriages the number of **stack-sortable permutations** is called the nth **Catalan number** C_n

Consider next 'Dyck graphs' like this (for $n = 4$)
There are $2n = 8$ segments to the graph
The graph never goes below the horizontal axis.
The slopes are always 45 degrees up or down.
The graph ends on the horizontal axis.

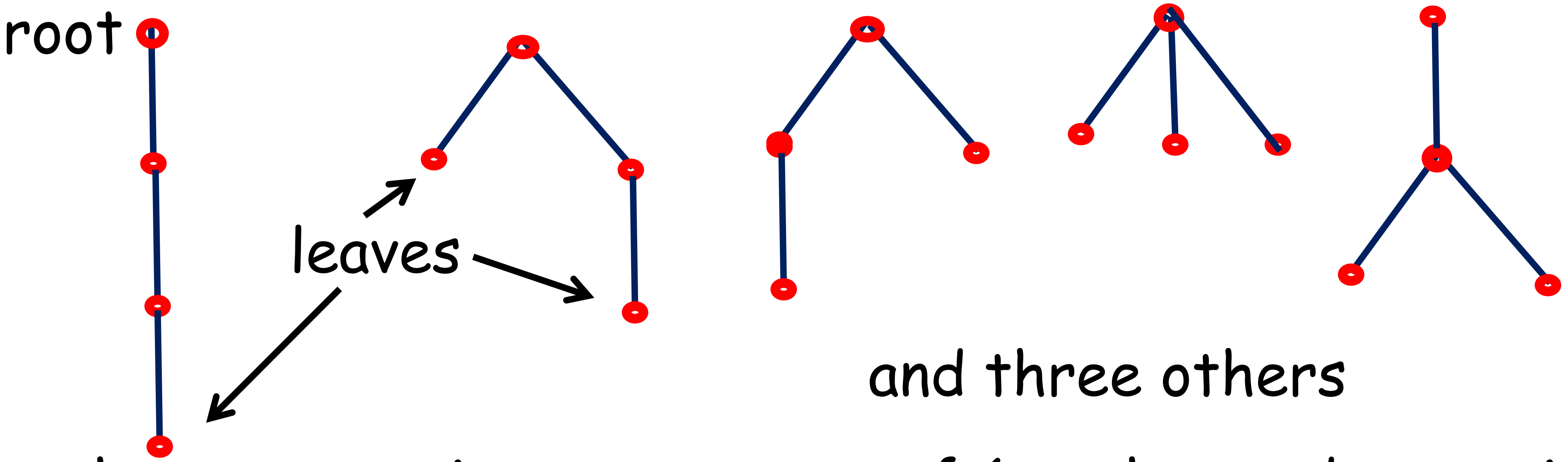


- What sequence of pushes and pops will this correspond to?
- What will be the resulting stack-sortable permutation?
- How many such 'Dyck graphs' are there for this n?
- Take the stack-sortable permutations 2143 and 4132 and draw the corresponding Dyck graphs.


push
push
pop
push
12
3
4
12
4
3
1
2
4
1
4
23
1
423
1
423
1423

Thirdly consider ‘rooted (planar) trees’. These are rather strange trees since the roots are at the top and the leaves at the bottom.

Start with 3 edges. The rooted trees with 3 edges are



and three others

Can you see how to associate a sequence of 6 pushes and pops with each of these? (Follow round the ‘outside’ of the tree.)
What rooted trees (4 edges) will correspond to the train carriage permutations 4132 and 2143 with 4 carriages?

Stack-sortable permutations of 1,2,...n
Dyck graphs with $2n$ edges
Rooted trees with n edges
(Also joining pairs of points starting with $2n$ points round a circle)

are all connected. The number of each, for a given n, is called the <u>Catalan number</u> C_n

$C_1 = 1$, $C_2 = 2$, $C_3 = 5$, $C_4 = 14$

How can we calculate C_n without a lot of bother?

Let's examine this stack sort with 8 numbers

12345678 $\longrightarrow$ 42138765

Notice that 8 goes to the fifth place.
Then the numbers 1,2,3,4 get their own stack sort
and the numbers 5,6,7 get their own stack sort
This breaks this stack sort into one with the first four numbers 1,2,3,4 and one with the next three numbers 5,6,7.

This always works and Catalan number C_8 can be worked out as

$$C_8 = C_0\,C_7 + C_1\,C_6 + C_2\,C_5 + C_3\,C_4 + C_4\,C_3 + C_5\,C_2 + C_6\,C_1 + C_7\,C_0$$

Here C_0 is defined to be 1.
There is a similar 'recurrence' formula for C_n

```
# Catalan numbers using the recurrence
# calculating  the Catalan numbers up  to C_N, counting
# from C_0 =1, C_1=1


N = int(input('what is the value of N?'))
cat=[1,1]
for i in range(1,N):
   cat.append(0)

for n in range(2,N+1):
    c=0
    for k in range(1,n+1):
       c=c+cat[k-1]*cat[n-k]
    cat[n]=c
print(cat)
```

It's easy then to write a python program to calculate Catalan numbers

what is the value of N? 20

(This output starts with C_0)

[1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796, 58786, 208012, 742900, 2674440, 9694845, 35357670, 129644790, 477638700, 1767263190, 6564120420]

There is an explicit formula for C_n which is a little more tricky to prove

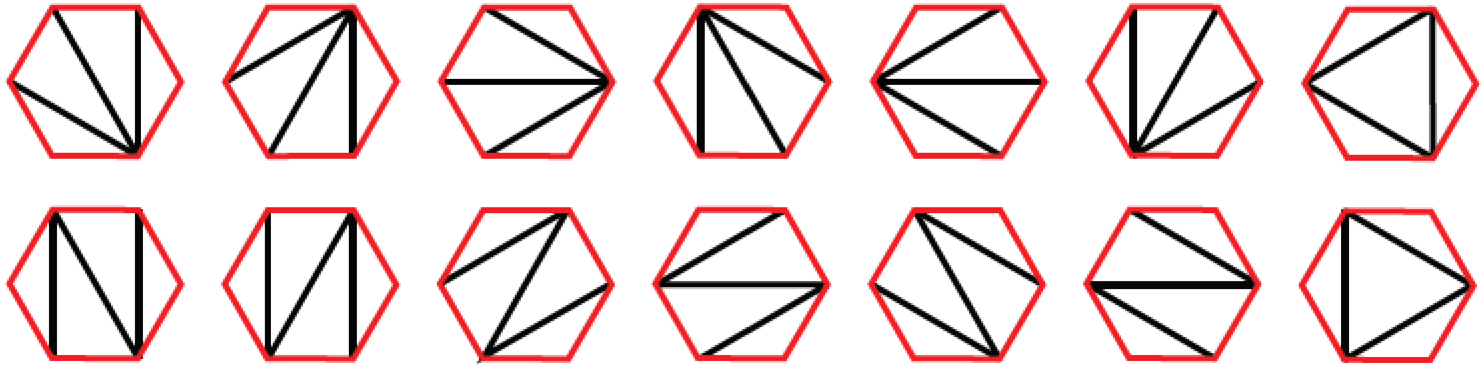
$$C_n = \frac{1}{n+1}\binom{2n}{n} = \frac{(2n)!}{(n+1)(n!)^2}.$$

$$C_4 = \frac{8.7.6.5.4.3.2.1}{5.1.2.3.4.1.2.3.4}$$

Can you cancel that down to 14 ?

Catalan numbers occur in many other contexts in maths and computer science. Find more on the internet, e.g. http://www.math.ucla.edu/~pak/lectures/Cat/pakcat.htm

For example C_n is the number of ways in which to split up a regular polygon with $n+2$ sides into triangles !



$C_4 = 14$

THANK YOU FOR YOUR ATTENTION

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