Remainders, consecutive whole numbers and card surprises

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Remainders

Take for example *remainders on division by 4*

For short I shall say *remainders mod 4*
and I shall write

$35 \equiv 3 \bmod 4$

$48 \equiv 0 \bmod 4$

Clearly all remainders are 0, 1, 2 or 3.

Mod 6 all remainders will be 0,1,2,3,4 or 5

Mod 2 all remainders will be
0 (even numbers) or 1 (odd numbers)

Suppose we have four consecutive whole numbers. Why will the remainders be, in some order, 0,1,2,3 ?

Eight consecutive numbers, why will the remainders be, in some order, 0,1,2,3,4,5,6,7 ?

When the remainders mod 8 are

0 or 4	0
1 or 5	1
2 or 6	2
3 or 7	3

what are the remainders mod 4?

Now we can get down to business !

Look at the jumbled lists of numbers 1,2,...,12:

6,5,7,8,4,3,9,2,10,1,11,12

5,6,4,3,7,8,2,9,10,11,1,12

Reading left to right can you see something they have in common which is not shared by

5,6,4,3,8,7,2,9,10,11,1,12 ?
(think: jumbled consecutive numbers)

I shall call a jumble ("permutation") of 1,2,...,12 (or more generally of 1,2,...,n) <u>good</u> if, reading left to right, <u>every set of numbers is a jumble of consecutive numbers</u>

e.g. 6,5,7,8,4,3,9,2,10,1,11,12

(and bad otherwise !
e.g. 5,6,4,3,8,7,2,9,10,11,1,12)

Can you find another two good jumbles (preferably not just 1,2,...,12 itself !)

Looking at your good jumbles of 1,2,...,12 what do you notice about the first four, the next four, and the last four, numbers in your jumble, thinking about remainders mod 4?

Now here is a cunning way to produce a good jumble. The numbers will be top to bottom rather than left to right. Think of them as numbered cards.

1
2
3
4
5
6
7
8
9
10
11
12

6	
7	
8	5
9	4
10	3
11	2
12	1

First, “deal off” some cards into a separate stack

6
5
4
7
3
8
9
10
2
11
12
1

Now insert the dealt off cards into the remaining ones, <u>in descending order</u>

Here is one possible way, you try another.

Hey presto! A good jumble! (Reading top to bottom now.)

Now look at the remainders mod 4 of

- the top four cards
- the next four cards
- the last four cards

What do you notice ?

I want to try to understand this !

6
5
4
7

3
8
9
10

2
11
12
1

The top 4 are consecutive so
remainders are all different mod 4

The top 8 are consecutive so the
remainders are all different mod 8

All remainders mod 8 are

0	4	grouped into
1	5	pairs with the
2	6	same remainder
3	7	mod 4

6
5
4
7

3
8
9
10

The top four numbers have used up one from each row
so the next four numbers must also come from four
different rows
and these all have different remainders mod 4

All 12 have different remainders mod 12

All remainders mod 12 are

0	4	8
1	5	9
2	6	10
3	7	11

grouped into triples with the same remainder mod 4

6
5
4
7

3
8
9
10

2
11
12
1

The top four use up one from each row, the next four use up a different one from each row.....
So the last four must use up the remaining one from each row
The last four have different remainders mod 4.

So every group of four has four different remainders mod 4

You could also divide the good jumble of 1,2,...,12
into <u>four groups of three</u>

The top three have different remainders mod 3
The next three have different remainders mod 3
The next three have different remianders mod 3
The last three have different remainders mod 3

or, of course <u>six groups of two</u>

6
5
4

7
3
8

9
10
2

11
12
1

In general, for a jumble of 1,2,...,*kn*, produced for example by dealing some cards one by one into a second pile and then inserting into the remaining ones in descending order, the cards in each group of k cards starting at the top have different remainders mod k.

This is called the **Gilbreath Principle.**

<u>Now let's apply this to packs of 52 cards.</u>

Take a pack of cards which are arranged in the order of repeating suits, say

♠ ♥ ♣ ♦ ♠ ♥ ♣ ♦ ♠ ♥ ♣ ♦ ...

This is like having remainders mod 4 in the repeating order 1,2,3,0,1,2,3,0,1,2,3,0.....,so like having 1,2,3,4,...,52 in order.

Now deal any number face down onto the table, making two piles, one undealt and one dealt. Roughly riffle the two piles together....
Every group of four from the top is four different remainders (suits)

The same works for ANY grouping, e.g.
red, black, red, black, red, black.... (26 times)

every group of two after dealing and rough riffling will be different colours

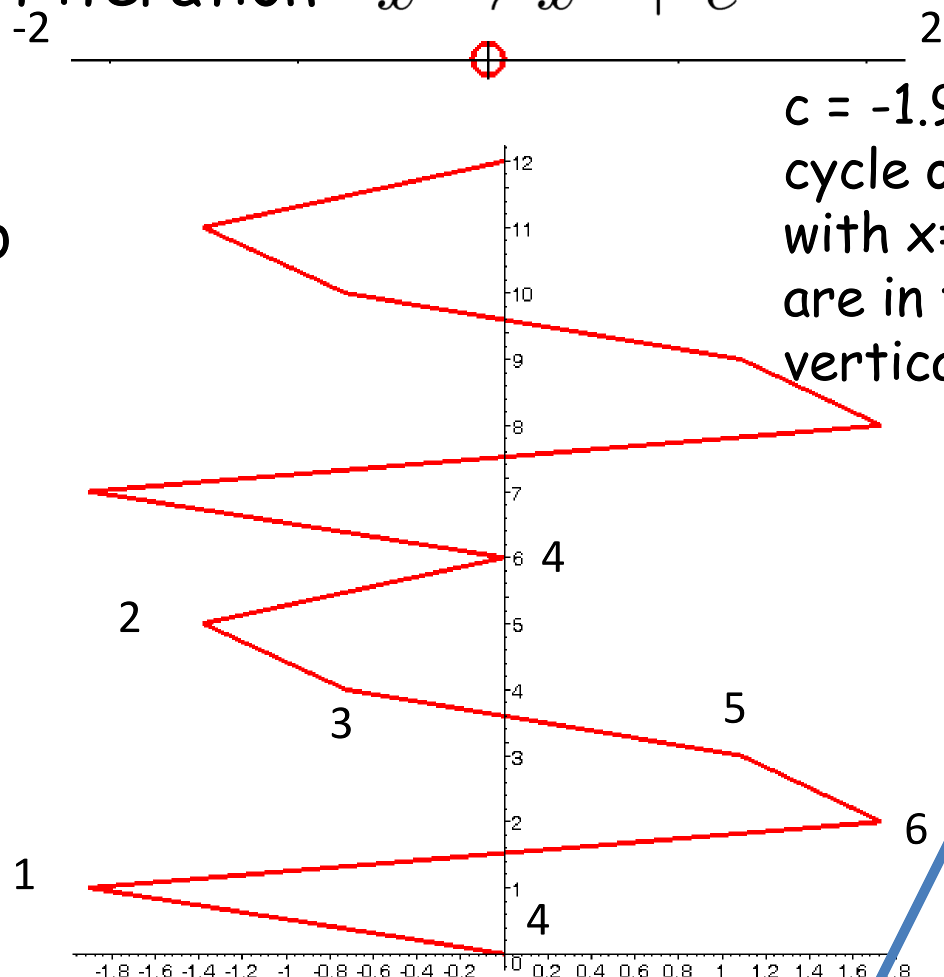
or

1,2,3,4,5,6,7,8,9,10,J,Q,K, 1,2,3,4,5,6,7,8,9,10,J,Q,K,... (4 times)

every group of thirteen after dealing and rough riffling will be different numbers.

There is also an extraordinary connection with the 'Mandelbrot iteration' $x \to x^2 + c$

Label the values of *x* from left to right: 1 for the leftmost, 2 for the next leftmost, etc. Then look at the downward-sloping lines, 4 to 2, 2 to 3 etc.

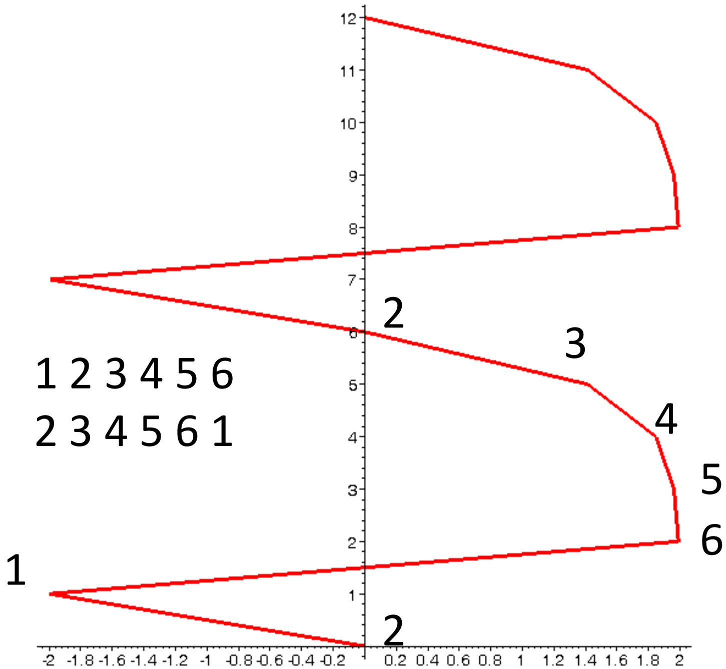


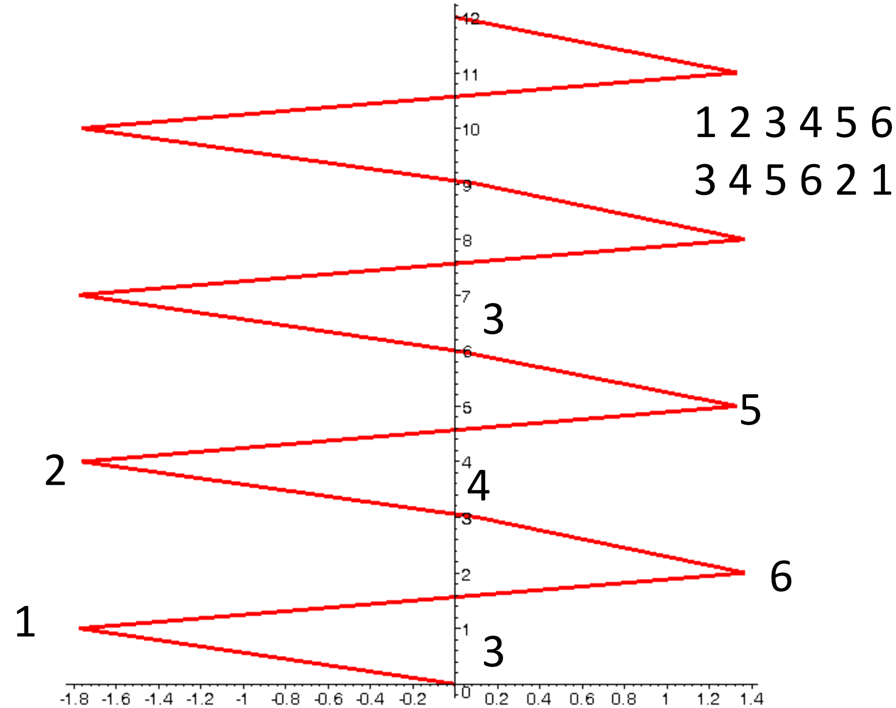
c = -1.907280091... gives a cycle of length 6. Starting with x=0 the successive x are in the diagram, on the vertical axis

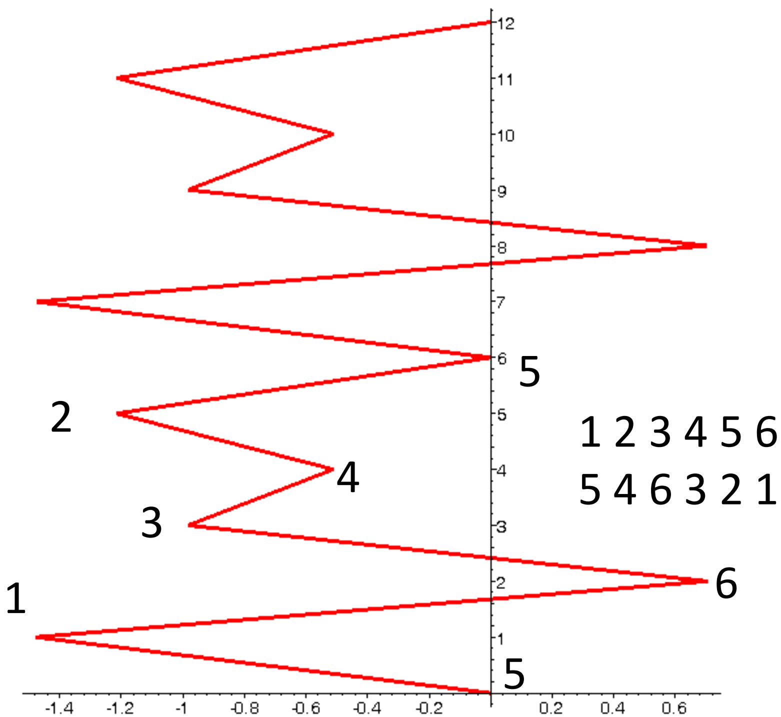
1	2	3	4	5	6
4	3	5	2	6	1

What do you notice about the lower line of numbers?

This happens for all cycles of finite length (result of Milnor-Thurston)







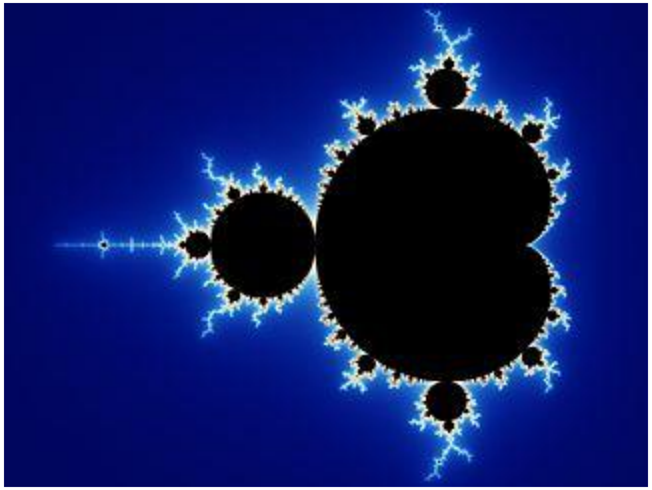
These are the remaining three 6-cycles arising from the 'square-and-add-c' construction. There are five such 6-cycles in all, and these give all the cyclic Gilbreath permutations of 6 numbers.

These stacks (sequences) of numbers where each block from the top down consists of consecutive numbers in some order arise in a completely different and most extraordinary context.

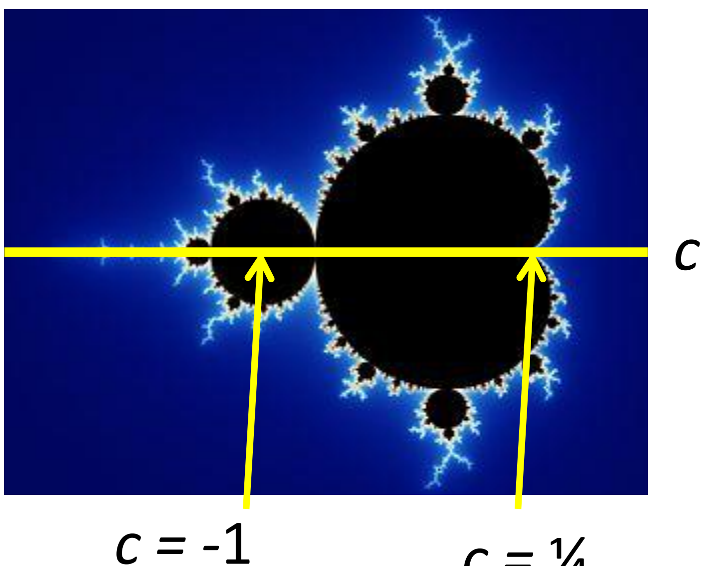
The connexion between the two was proved a while ago by two great contemporary mathematicians John Milnor and William Thurston

Milnor recently received the Abel Prize 2011 which is like a Nobel Prize for Mathematics

Thurston



This is actually a picture in the *complex plane* but ordinary real numbers *c* lie along the middle line...



Those numbers *c* within the black shape are those for which the values of *x* under 'start with 0, square and add *c*' stay **bounded.** (That is don't wander off to infinity.)

You can take *c* to be a 'complex number' $c = a + b\sqrt{-1}$ too and look at the sequence starting at 0.