Coxeter number friezes

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Start with a row of 0's followed by a row of 1's. Start the third row with a suitable string of positive integers. Then construct subsequent rows by ensuring that each tiny diamond

$$\begin{array}{ccc} & b & \\ a & & d \\ & c & \end{array}$$

of the pattern obeys the so-called **unimodular rule** $ad - bc = 1$. As this can be rearranged to give $c = (ad - 1)/b$, all the numbers in lower rows of the pattern can be calculated. For example, if we start the third row with the string of numbers 1 2 2 3 1 2 4, we get the pattern

$$\begin{array}{ccccccccccccccccccc}
0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 \\
& 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & \\
& & 1 & & 2 & & 2 & & 3 & & 1 & & 2 & & 4 & & & & \\
& & & 1 & & 3 & & 5 & & 2 & & 1 & & 7 & & & & & \\
& & & & 1 & & 7 & & 3 & & 1 & & 3 & & & & & & \\
& & & & & 2 & & 4 & & 1 & & 2 & & & & & & & \\
& & & & & & 1 & & 1 & & 1 & & & & & & & & \\
& & & & & & & 0 & & 0 & & & & & & & & &
\end{array}$$

Having reached a row of 1's followed by a row of 0's, it is natural to continue these two lowest rows to the right. We can then use the unimodular rule to calculate more terms:

0		0		0		0		0		0		0		0		0		0			
	1		1		1		1		1		1		1		1		1				
		1		2		2		3		1		2		4		1					
			1		3		5		2		1		7		3		1				
				1		7		3		1		3		5		2		1			
					2		4		1		2		2		3		1		2		
						1		1		1		1		1		1		1		1	
							0		0		0		0		0		0		0		0

Understanding periodicity

Consider a small piece of a frieze in which the unimodular rule holds:

$$\begin{array}{ccccc} & & c & & \\ & b & & r & \\ a & & q & & \\ & p & & & \end{array}$$

The rule shows that $br - cq = 1$ and $aq - bp = 1$. So

$$\begin{array}{rrcl} & br - cq & = & aq - bp. \\ \text{This gives} & b(p + r) & = & q(a + c), \\ \text{so that} & \dfrac{p + r}{q} & = & \dfrac{a + c}{b} \end{array}$$

Let's call this equation the *ratios rule*. It also holds for triples that run north-west to south-east,

$$
\begin{array}{cccccccccccccccccccc}
0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & \\
& 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 \\
& & x & & y & & c & & . & & . & & . & & . & & x'' & & y'' & \\
& & & z & & b & & r & & . & & . & & . & & . & & . & & \\
& & & & a & & q & & . & & . & & . & & . & & . & & & \\
& & & & & p & & . & & x' & & y' & & . & & . & & & & \\
& & & & & & 1 & & 1 & & 1 & & 1 & & 1 & & & & & \\
& & & & & & & 0 & & 0 & & 0 & & 0 & & 0 & & & &
\end{array}
$$

Applying the ratios rule to consecutive triples on the left gives

$$\frac{x+0}{1} = \frac{z+1}{y} = \frac{a+c}{b} = \frac{p+r}{q} = \ldots = \frac{0+x'}{1},$$

so $x = x'$. Similarly $(x'+0)/1 = (0+x'')/1$, so $x = x''$ and also $y = y' = y''$. This shows that the pattern is repeating and it is easy to check that, if the frieze has $n+1$ rows, it repeats n units to the right. The fact that $x = x'$ and that the pattern can be built from the bottom upwards shows that the pattern has glide-reflection symmetry as well.

How to ensure that a frieze consists of whole numbers Method 1: Diagonal entries

A complete frieze with $n+1$ rows is determined as soon as we have chosen the $n+1$ terms in a diagonal. How much freedom of choice do we have if we wish all the numbers to be integers? Suppose that the frieze is:

$$\begin{array}{ccccccccccccccc}
0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & \dots \\
 & f_0 & & 1 & & 1 & & 1 & & 1 & & 1 & & \dots & \\
 & & f_1 & & a_1 & & a_2 & & a_3 & & a_4 & & \dots & & \\
 & & & f_2 & & . & & . & & . & & \dots & & & \\
 & & & & f_3 & & . & & . & & \dots & & & & \\
 & & & & & f_4 & & . & & \dots & & & & & \\
 & & & & & & \ddots & & & & & & & &
\end{array}$$

By the ratios rule,

$$\frac{f_0+f_2}{f_1} = \frac{0+a_1}{1} = a_1, \quad \frac{f_1+f_3}{f_2} = a_2, \quad \frac{f_2+f_4}{f_3} = a_3, \quad \text{etc.}$$

So in a frieze of integers each f_i divides the sum, $f_{i-1}+f_{i+1}$, of its neighbours.

Converely, if this divisibility rule is satisfied in some diagonal of integers, then all the a_i's are integers. Suppose that the next diagonal consists of

$$g_0(= 0) \quad g_1(= 1) \quad g_2(= a_1) \quad g_3 \quad g_4 \quad \dots$$

By the ratios rule,

$$\frac{g_{i-1} + g_{i+1}}{g_i} = \frac{f_{i-1} + f_{i+1}}{f_i} = a_i.$$

So $g_{i-1} + g_{i+1} = a_i g_i$. Hence $g_{i+1} = a_i g_i - g_{i-1}$. This tells us that if g_{i-1} and g_i are integers, then so is g_{i+1}. But both g_0 and g_1 are integers, so g_2 is also an integer. Repeating the argument, $g_3, g_4, \dots$ must all be integers. The ratios rule and the divisibility property of the f_i's show that the g_i's also have the divisibility property. Thus, starting with a single diagonal of integers that have the divisibility property, the next diagonal must also have these properties ... and so must the next diagonal and all the diagonals, so the entire frieze consists of integers.

How to ensure that a frieze consists of whole numbers
Method 2: Triangulated polygons

To find suitable numbers for the third row of a frieze with $n+1$ rows, triangulate a convex n-gon and note how many triangles meet at each vertex. These numbers, repeated in cyclic order, produce a frieze of integers. For example, when $n = 7$, a triangulated heptagon gives the cycle of numbers 1 2 2 3 1 2 4 that form the third row of our first frieze. (See diagram.)

John Conway proved that this recipe always produces a frieze of natural numbers and, conversely, that every frieze of natural numbers comes from a triangulated polygon in this way.

One step in Conway's proof: Construct a frieze for a triangulated octagon from our original (heptagonal) frieze. Separate the triangular portions of our original frieze to leave diagonal channels between them. Then put a single line of new numbers into each channel so that each new number is the sum of its nearest neighbours in the separated portions. This gives the new pattern shown:

0 0 0 0 0 0 0 0 0 0

1 1 1 1 1 1 1 **1** **1**

1 **2** 2 2 3 1 2 **5** 1 **2**

1 **3** 3 5 2 1 **9** 4 1 **3**

1 **4** 7 3 1 **4** 7 3 1 4

1 **9** 4 1 **3** 3 5 2 1 **9**

2 **5** 1 **2** 2 2 3 1 2 **5**

1 **1** **1** 1 1 1 1 1 1

0 0 0 0 0 0 0 0

This new pattern is a frieze, because when we pull apart a tiny unimodular diamond

$$\begin{array}{ccc} & b & \\ a & & d \\ & c & \end{array}$$

and insert the entries $a + b$ and $c + d$ in the channel as shown,

$$\begin{array}{cccc} & & b & \\ & a+b & & d \\ a & & c+d & \\ & c & & \end{array}$$

the unimodular law still holds in the two diamonds created, since

$$(a + b)d - b(c + d) = a(c + d) - (a + b)c = ad - bc = 1.$$

We can check that the third row of our new frieze corresponds to the cycle of numbers of triangles that meet at each vertex of the octagon.