Iteration and Geometrical Fractals

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1 Iteration

Definition 1.1. Iteration is a a repetition of a mathematical or computational procedures applied to the result of a previous application.

Remember that solutions to equations of the form ax^2+bx+c, where $a \neq 0$ and $b^2 > 4ac$, can be written as:

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

Infinitely Nested Radicals: (Appendix A)

- Solve $x = \sqrt{1+\sqrt{1+...}} = \phi$ (The number ϕ is often referred to as the *golden ratio*)
- Solve $x = \sqrt{2+\sqrt{2+...}}$
- Solve $x = \sqrt{a+b\sqrt{a+...}}$
- Find an equation satisfied by x if $x = \sqrt[n]{a+b\sqrt[n]{a+...}}$
- Write $x = 4$ as an infinitely nested radical.
- Given that $e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + ...$ show that $x^{e-2} = \sqrt{x\sqrt[3]{x\sqrt[4]{x...}}}$
  (Note: $n! = n \times (n-1) \times ... \times 2 \times 1$)

Continued Fractions: (Appendix B)

- Solve $x = 1 + \frac{1}{1+\frac{1}{...}}$
- Solve $x = 1 + \frac{1}{2+\frac{1}{2+...}}$
- In general $x = a + \frac{b}{a+\frac{b}{a+...}}$
- Fold up the continued fraction $x = [0; 1, 1, 3, 2]$ to show that $x = \frac{9}{16}$
  The notation $x = [a; b, c, ...]$ means $x = a + \frac{1}{b+\frac{1}{c+\frac{1}{...}}}$
- Write the the ratio os successive numbers in the Fibonacci sequence as continued fractions. What happens?
  Fibonacci Sequence: $1, 1, 2, 3, 5, 8...$ ($u_n = u_{n-1} + u_{n-2}$)
- Write the ratio of successive square numbers as continued fractions. What happens?

Are there any numbers which may not be written in either of these ways?

1.1 Geometrical Fractals

Definition 1.2. A fractal is a never-ending pattern. Fractals are infinitely complex patterns that are self-similar across different scales. They are created by repeating a simple process over and over in an ongoing feedback loop. Driven by recursion, fractals are images of dynamic systems the pictures of Chaos. - The Fractal Foundation

Read all the instructions before starting. Creating a *Koch Snowflake*:

1. Use a given prefabricated equilateral triangle to draw its outline on a blank piece of paper.
2. Record the state of the current drawing into a table like the one below.
3. Divide each edge of the given triangle by 3 to make a smaller equilateral triangles.
4. Use one of these smaller triangles to drawn a small triangle to each edge of the current drawing.
5. Repeat the process from step 2 with one of the smaller triangles.

Level	No. triangles added	No. edges	Length of an edge	Total Perimeter	Total area
0	0	3	L	3L	A
1					

Ask yourselves these following questions:

- What happens to length of an edge at each step?
- What happen to the area of a triangle at each level?
- What happens to the total perimeter at each step?
- What happens to the total area at each step?

Can you work out the perimeter of the shape at the n^{th} level? What about the area? (Appendix C)

Coastline Paradox.

Definition 1.3. Determining the length of a country's coastline is not as simple as it first appears, as first considered by L. F. Richardson (1881-1953) and sometimes known as the Richardson effect (Mandelbrot 1983, p. 28). In fact, the answer depends on the length of the ruler you use for the measurements. - http://mathworld.wolfram.com/CoastlineParadox.html (adapted).

List down any fractals you can think of from nature i.e. coastlines.
Why do fractals appear in nature?

A Transforming the Nested Radical form

The general infinite nested radical form can be solved by rearranging,

$$\begin{aligned} x &= \sqrt[n]{a + b\sqrt[n]{a + \ldots}} \\ x^n &= a + b\sqrt[n]{a + \ldots} \\ &= a + bx \\ x^n - bx - a &= 0 \end{aligned}$$

In the case of $n = 2$ the equation can be solved for x using the quadratic formula,

$$x = \frac{b + \sqrt{b^2 + 4a}}{2}$$

Thus, if the nested radicals has $a = b = 1$ we have $x = \frac{1+\sqrt{5}}{2}$ and we may disregard the negative because x was the positive radical.
Equally if we had $a = 2, b = 1$, then we have $x = 2$.

The case of $x = 4$ is really about doing the same, but in reverse order. Luckily we have the case for $x = 2$ so we may just double this:

$$\begin{aligned} x &= 2 \\ &= \frac{1 + \sqrt{1^2 + 4 \times 2}}{2} \\ \Rightarrow y &= 4 = x \times 2 \\ &= 2 \times \frac{1 + \sqrt{1^2 + 4 \times 1}}{2} \\ &= \frac{2 + 2\sqrt{1^2 + 4 \times 1}}{2} \\ &= \frac{2 + \sqrt{2^2 \times 1 + 2^2 \times 4 \times 1}}{2} \\ &= \frac{2 + \sqrt{\times 2^2 + 4 \times 4}}{2} \end{aligned}$$

By comparing this last line with the formula we have for nested radicals we can see that in case of $y = 4$ the parameters are $a = 4, b = 2$. Now that he path is clear, try to make some other number i.e. what is the nested radical for $x = \sqrt{2}$? Are there any numbers which you cannot make?

The last result given in the problem set is very interesting. It happens that the number e is like the number π in that it has an infinitely long decimal expansion which is not repeating and unpredictable. Yet it crops up all the time! This is just one example of where it unexpectedly shows itself.

If $e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \ldots$ it is not hard to see through some factorising that $e = 1 + 1 + \frac{1}{2}(1 + \frac{1}{3}(1 + \frac{1}{4}(1 + \frac{1}{5}(1 + \ldots))))$. Using this form of the expression, in conjunction with the laws of exponents, it is possible to get the required result quite quickly:

$$\begin{aligned} x^{e-2} &= x^{\frac{1}{2}(1+\frac{1}{3}(1+\frac{1}{4}(1+\frac{1}{5}(1+\ldots))))} \\ &= (x(x(x\ldots)^{1/4})^{1/3})^{1/2} \\ &= \sqrt{x\sqrt[3]{x\sqrt[4]{x\ldots}}} \end{aligned}$$

B Writing numbers as continued fraction

The first three problems are solved by recognising the self-similarity in the right hand side:

$$\begin{aligned}
x &= a+\cfrac{b}{a+\frac{b}{a+...}} \\
x &= a+\frac{b}{x} \\
x^2 &= ax+b \\
x^2-ax-b &= 0 \\
x &= \frac{a+\sqrt{a^2+4b}}{2}
\end{aligned}$$

Therefore, for the first question asked we must just substitute the parameters $a=b=1 \Rightarrow x=\frac{1+\sqrt{5}}{2}$. However, because the second question is not quite as regular, the argument has to be run through again, but with the correct numbers it happens that it becomes $x=\sqrt{2}$. It is rather interesting how changing one single number in the set-up makes a huge difference in the ease of computation.

The next question is very similar; it again focuses on simplifying the continued fraction.

$$\begin{aligned}
\frac{9}{16} &= [0;1,1,3,2] \\
x &= 0+\cfrac{1}{1+\frac{1}{1+\frac{1}{3+\frac{1}{2}}}} \\
&= 0+\cfrac{1}{1+\frac{1}{1+\frac{1}{\frac{7}{2}}}} \\
&= 0+\cfrac{1}{1+\frac{7}{9}} \\
&= 0+\cfrac{1}{\frac{16}{9}} \\
&= \frac{9}{16}
\end{aligned}$$

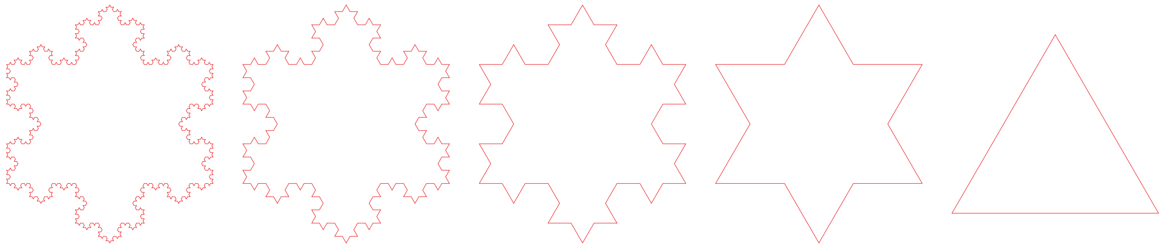
Using this same process with the appropriate numbers it is not hard to compute the the ratio of successive Fibonacci terms:

$$\begin{aligned}
\frac{3}{2} &= [1;2] \\
\frac{5}{3} &= [1;1,2] \\
\frac{u_{n+1}}{u_n} &= [1;1,...,1,2]
\end{aligned}$$

where there are an $n-2$ number of 1s preceding the 2. This is significant because we saw that $\phi=[1,1,1,1...]$ which tells us that $n\to\infty$ we have $\frac{u_{n+1}}{u_n}\to\phi$ or that is to say the ration of successive numbers in the Fibonacci sequence tend to the golden ratio.

C Koch Snowflake

The created images should look something like these:



Level	No. triangles added	Number of edges	Length of an edge	Total Perimeter	Total Area
0	0	3	L	3L	A
1	3	12	$\frac{1}{3}$L	4L	$\frac{4}{3}$A
2	12	48	$\frac{1}{9}$L	$\frac{16}{3}$L	$\frac{40}{27}$A
3	48	192	$\frac{1}{27}$L	$\frac{64}{9}$L	$\frac{376}{243}$A
n	$3 \times 4^{n-1}$	3×4^n	$\frac{1}{3^n}$L	$3\frac{4^n}{3^n}$L	$A_{n-1} + \frac{1}{3}(\frac{4}{9})^{n-1}$A

This last line is somewhat tricky to see where it has come from so we will break it down.

Let N_n be the number of edges; L_n be the length of a single edge; P_n be the length of the perimeter; and A_n the snowflake's area after the n^{th} iteration. Note that we denote the area of the initial $n = 0$ triangle as A, and the length of an initial $n = 0$ side as L. Then,

$$
\begin{aligned}
N_n &= 3 \times 4^n \\
L_n &= \frac{1}{3}^n \\
P_n &= N_n \times L_n \\
&= 3(\frac{4}{3})^n \\
A_n &= A_{n-1} + \frac{1}{4} N_n L_n^2 A \\
&= A_{n-1} + \frac{1}{3}(\frac{4}{9})^{n-1} A
\end{aligned}
$$

The first two are taken to be fairly obvious. From which it is clear that the total perimeter is the number of edges times the length of an edge, hence the formula for P_n.

The formula for A_n is more complicated. The important features is that the second term is simply adding the new area which is found by taking the total number of added triangles' area. The added triangles is a quarter the number of edges and using the area scale factor, which is the square of the edge scale factor, we get their contribution to the total area.

Interestingly, the area recurrence relation can be solved to give:

$$A_n = \frac{1}{5}\left[8 - 3(\frac{4}{9})\right] A$$

so as $n \to \infty$,

$$A_\infty = \frac{8}{5}A$$

Which is to say that the area is always finite. Yet, it is not hard to see that $P_n \to \infty$ as $n \to \infty$ being a simple geometric sequence. So the Koch snowflake has an infinite boundary, but finite area.

There are many fractals that appear in nature. Here is a list of just a few:

- Coastlines
- Branches on a tree.
- Frost/Snow Flakes
- Broccoli esp. Romanesco broccoli
- Rivers
- High voltage breakdown (Lichtenberg figure)
- The Universe.
- Mountains (Used to save processing in computer science)

Fractals are so common because of an idea by the Greek philosopher Democritus - The Atomic Theory. If we agree with the assumption that world is made by tiny things which group together, then it makes sense that those groups have similar properties to the things that make them up and so on. Such that if one were to scale up from the true atom, rather than scale down from our world view, then the *fractal geometry of nature* becomes consequential.

In the traditional India belief system, from which things like "Karma" come from, there is a *cosmic law* which states "As within - as without" - the idea of a fractal millennia before the mathematics of it.

The reader may wish to pursue this topic further by asking what happens if squares are used instead? Or if rather than adding triangles, they are taken away. There are many online resources which allows the reader to entertain themselves within the topic of geometrical fractals, for example:

`http://www.maths.liv.ac.uk/~mathsclub/talks/20011027/talk1/extras/index.html}`

is a very interesting applet.

In terms of the iterations questions, there are many further which could be investigated here are a couple:

- Using $(x+1)^2 - 1 = x(x+2)$ show that $3 = \sqrt{1 + 2\sqrt{1 + 3\sqrt{1 + 4\sqrt{1 + 5\sqrt{1 + ...}}}}}$
- Prove that if a simple continued fraction is periodic, then its value is a quadratic irrational.