University of Liverpool Maths Club

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COMPLETING MAGIC SQUARES

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First, a 4x4 magic square to remind you what it is:

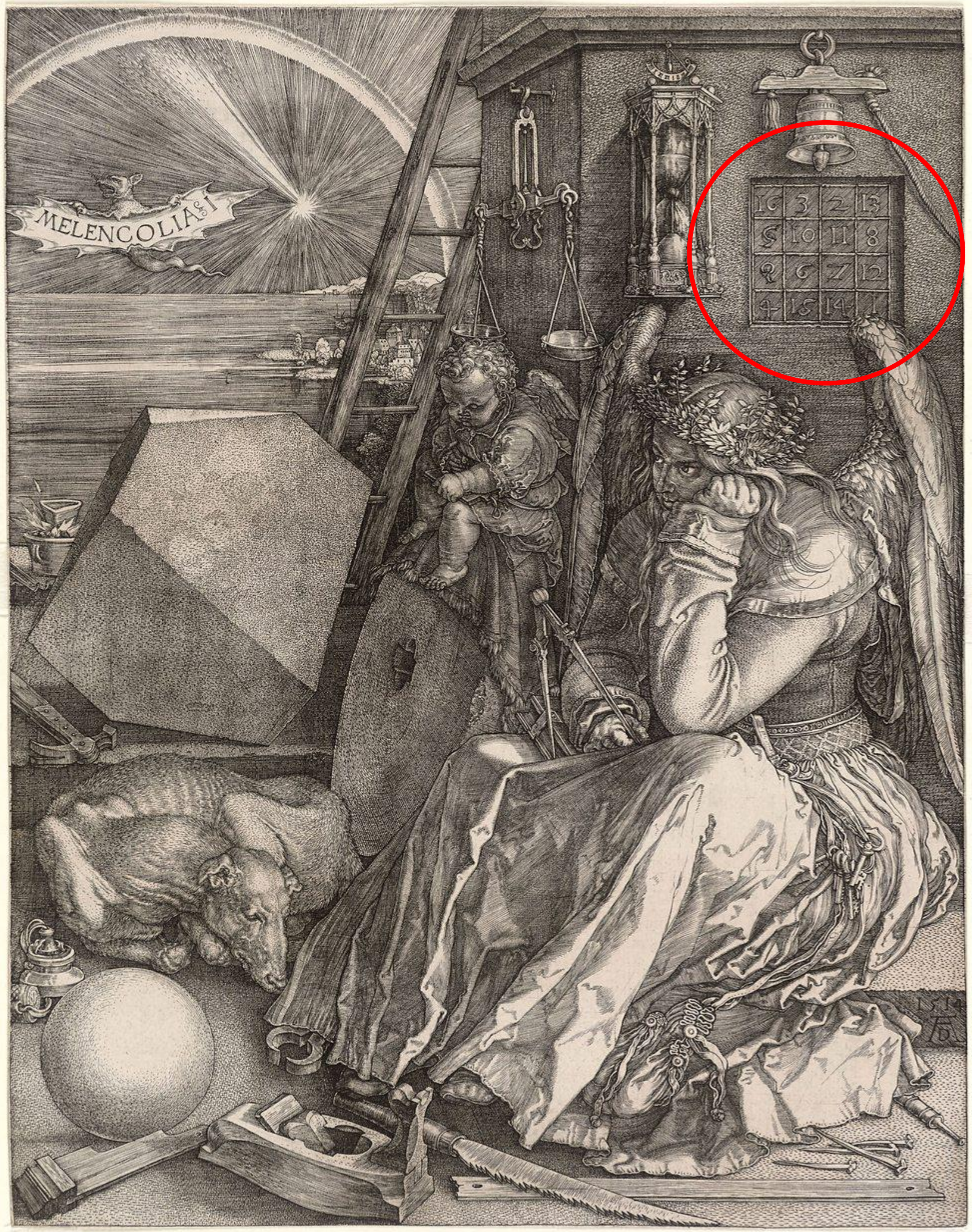
8	11	14	1
13	2	7	12
3	16	9	6
10	5	4	15

Every row, column and the main diagonals add to the same total (here it is 34).

What other groups of numbers add to 34?

Actually in this example all broken diagonals also add to 34, and indeed 2x2 blocks do too.

Furthermore the numbers 1,2,...16 appear just once, and 1+2+3+...+16=136=4x34 so each sum must be 34.



16	3	2	13
5	10	11	8
9	6	7	12
4	15	14	1

Albrecht Dürer’s engraving *Melencolia I*, celebrating its 500^{th} birthday this year.

The rows, columns and main diagonals all sum to 34. What about broken diagonals and 2x2 boxes?

	4		7
6		1	
		8	13
16	5		

This magic square contains the numbers 1 to 16 and every row, column and the two main diagonals add to 34. Can you fill in the missing entries?

Do the broken diagonals add to 34 as well?
What about the 2x2 boxes?

	14	15	
9			12
	11	10	
16			

This one is a tiny bit harder to complete. Again you are only allowed to use rows, columns and the two main diagonals, each of which adds to 34. (And all entries are positive whole numbers!)

There are actually two solutions here. Can you find one or both of them?

(And in fact the numbers 1,2,...,16 appear once, though this is not needed to complete the square.)

For a while now we'll consider a smaller size, a 3 x 3 square where we assume that **all rows, columns and the two main diagonals add to the same sum, *s* say**. I shall **not assume** that the numbers 1,2,...,9 occur just once. The numbers a, b,...,k could be **any positive whole numbers (i.e. 1, 2, 3, 4, ...), and repetitions are allowed**

a	b	c
d	e	f
g	h	k

1. Can you fill in the rest of the square given a=4, b=5, s=18? Is there just one answer?
2. How about a=4, e=5, s=15? (How many answers? How many avoid repeating the same number?)
3. How about a=4, e=5, s=18?
4. Adding up the four lines through e gives 4s. Why is this equal to 3s + 3e? What follows from this?

a	b	c
d	e	f
g	h	k

Sum of all rows, columns and the two main diagonals is s, which we'll write as $3m$. Then $e = \frac{1}{3}s = m$.

Suppose given a, b and $s = 3m$ say.
Fill in the rest of the numbers.
So $e = m$, $c = 3m - a - b$ for example.

a	b	$3m - a - b$
$4m - 2a - b$	m	$2a+b - 2m$
$a+b - m$	$2m - b$	$2m - a$

For all the entries to be > 0 we need

$m < a + b < 3m$

$2m < 2a + b < 4m$

$a < 2m$

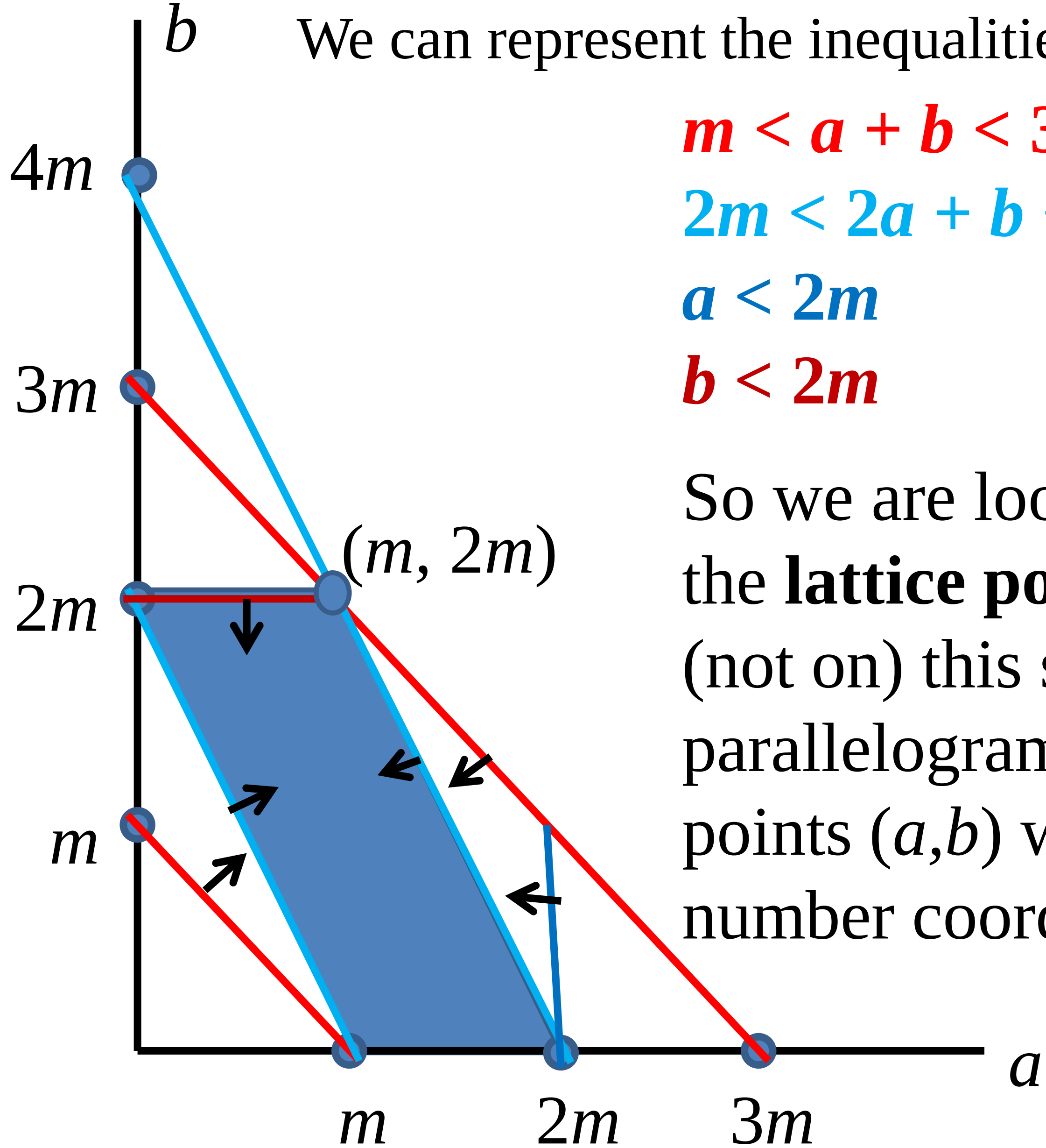
$b < 2m$

Suppose given only m, where the sum of each row, column and main diagonal is $s = 3m$.

How many solutions for a and b will there be making all the entries > 0?

For example, take $m = 2$, $m = 3$ in turn.

We can represent the inequalities on a diagram:



$m < a + b < 3m$

$2m < 2a + b < 4m$

$a < 2m$

$b < 2m$

So we are looking for all the **lattice points** inside (not on) this shaded parallelogram, that is points (a,b) with whole number coordinates.

How can we count this number of lattice points, that is the number of solutions of the 3x3 magic square when we are given just m?

There is a marvellous theorem called **Pick's Theorem** which says

Area of parallelogram = (Number of lattice points *inside*) + (half the number of lattice points *on the boundary*) – 1

Challenge Can you use this to find the number of lattice points inside the parallelogram, in terms of *m* ?

Pick's theorem works for any polygon whose corners are lattice points (integer coordinates)

From now on we'll consider 4x4 squares but again drop the requirement that the numbers 1, 2, ..., 16 should be used once each.

We just require all rows, columns and the two main diagonals to add to the same *magic total.*

a	b	c	d
$c+3x$	$d-3x$		
	$c-2x$		
$b-x$			$c+x$

Can you fill in the rest so that all sums of rows, columns and the two <u>main</u> diagonals equal the magic total $a+b+c+d$?

Do the broken diagonals or 2x2 blocks add to the same?

12	14		
		6	
	29		

Can you use the above diagram to complete the magic square on the left, with magic total 62?

3			
		9	
			10
	19		

Can you use the same method to fill in the blanks to find a magic square with magic total 51?

1			
	5		
		20	3

Can you use the same method to fill in the blanks to find magic squares with magic total 52?

Actually, if all rows, columns and diagonals including the broken diagonals of a 4x4 magic square add to 34 then it **follows that** the 2x2 boxes all add to 34 as well.

(Here, 34 can be replaced by any constant *k*.)

That’s another challenge to try at home.

Notes on the problems

Slide 5

a	14	15	*b*
9			12
	11	10	
16			

a+*b*=5 (from *s*=34) so there are four choices, (*a*,*b*)=(1,4),(4,1),(2,3),(3,2). But only (4,1) and (3.2) lead to solutions.

4	14	15	1
9	7	6	12
5	11	10	8
16	2	3	13

3	14	15	2
9	8	5	12
6	11	10	7
16	1	4	13

As it happens, both of these use the numbers 1,2,...,16 once but broken diagonals and 2x2 boxes don’t add to 34.

Slide 6,Qu.1

4	5	9
11	6	1
3	7	8

is the only solution

Slide 6,Qu.2

s=15

4	b	
	5	

any of b=3,4,5,6,7,8,9 leads to a valid solution. The only ones which avoid repeating numbers are b=3, b=9:

4	3	8
9	5	1
2	7	6

4	9	2
3	5	7
8	1	6

Slide 6,Qu.3: there are no solutions.

Qu.4: adding up the 4 lines through e gives $4s$ since each line adds to s.
It also counts all nine entries in the grid once, except for e which it counts 4 times. Since the total of the nine entries is $3s$ this means that
$4s = 3s + 3e$, and solving gives $e = \frac{1}{3} s$. That is the middle entry is always one-third of the magic total s. That is why Qu.3 has no solution.
It also means that for whole number entries, s must be a multiple of 3.

Slide 7 is answered on slide 8.

Slide 8: when $m=2$ we have five solutions $(a,b) = (1,3)$,$(2,1)$, $(2,2)$, $(2,3)$, $(3,1)$.
When $m = 3$ there are 13 solutions for (a,b).

Slide 9 just draws the inequalities for a,b on the (a,b) plane. The regions which satisfies all the inequalities is shaded.

Slide 10: Using Pick's formula the number of lattice points inside, which is the number of solutions (a,b), is $2m^2-2m+1$.

Slide 12

For the given square, take x = -2, c=25, d=11

a	b	c	d
$c+3x$	d-3x	$a+3x$	b-3x
d-2x	c-2x	$b+2x$	$a+2x$
b-x	$a+5x$	d-5x	$c+x$

12	14	25	11
19	7	6	20
15	29	10	8
16	2	21	23

Slide 13: here you can't use the diagram (above left) directly but if you rotate it by 90 degrees you can find suitable a,b,c,d,x and you get the following:

3	14	13	21
22	12	9	8
11	6	24	10
15	19	5	12

14	9	13	16
1	28	12	11
18	5	7	22
19	10	20	3