# Bayes' Theorem and Conditional Probability in the Real World

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# I Toss a Coin

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- What is the probabilty that it comes up heads?
- The coin comes up heads on the first toss. What is the probability that the next toss is also heads?

# Some Definitions

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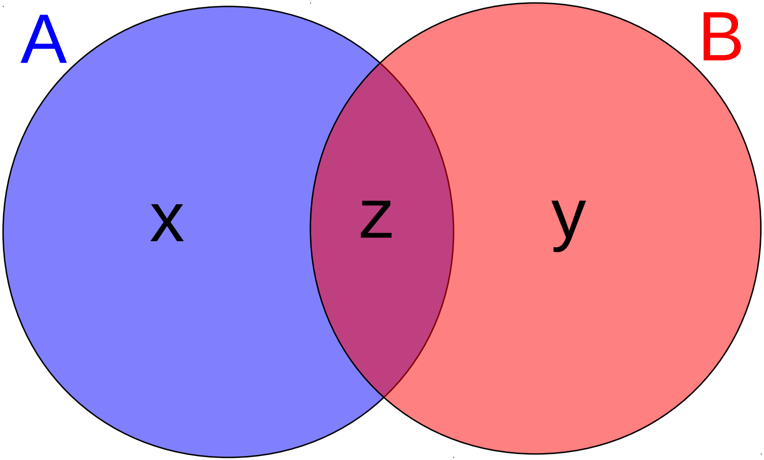
## Conditional

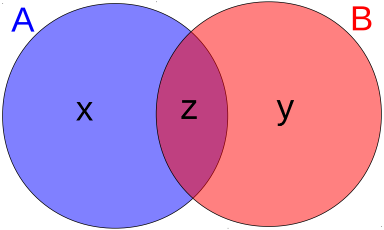
$$\mathbb{P}(A|B)$$

"Probability of A given B "

# Venn Diagram



# Venn Diagram - ctd




Can you write the following in terms of x, y and z:

- $\mathbb{P}(A \cap B)$?
- $\mathbb{P}(A)$? $\mathbb{P}(B)$?
- $\mathbb{P}(A|B)$?

Can you write $\mathbb{P}(A|B)$ in terms of the other probabilities?

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$$\mathbb{P}(A \cap B) = \mathbb{P}(A|B)\mathbb{P}(B) = \mathbb{P}(B|A)\mathbb{P}(A)$$

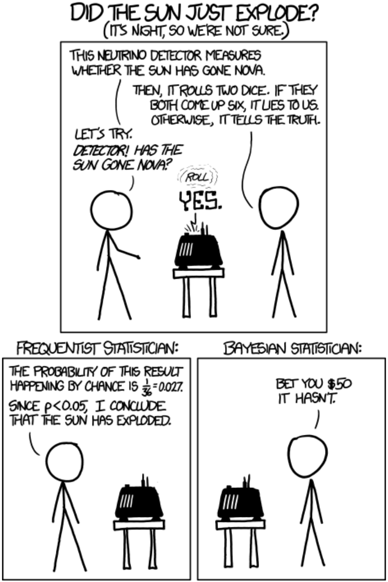
# Bayes' Theorem

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(B|A)\mathbb{P}(A)}{\mathbb{P}(B)}$$

$\mathbb{P}(B)$ is known as the Prior of B

$\mathbb{P}(B|A)$ is known as the Posterior

# Choice of Prior



# An Example

There are $50$ people in my lab, of whom $7$ are statisiticans.
Overall, only $\frac{1}{5}$ of my lab drink coffee. However, all but one of the statisicians drinks coffee.
You meet someone from my lab, and they are drinking coffee. What is the probability that they are a statistician?

# Solution

Write $\mathcal{S}$ for the event that someone is a statistician, and $\mathcal{C}$ for the event that they drink coffee.

$$\mathbb{P}(\mathcal{S}) = \frac{7}{50}$$
$$\mathbb{P}(\mathcal{C}) = \frac{1}{5}$$
$$\mathbb{P}(\mathcal{C}|\mathcal{S}) = \frac{6}{7}$$

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Hence:

$$\begin{aligned}\mathbb{P}(\mathcal{S}|\mathcal{C}) &= \frac{\mathbb{P}(\mathcal{C}|\mathcal{S})\mathbb{P}(\mathcal{S})}{\mathbb{P}(\mathcal{C})} \\ &= \frac{\frac{6}{7} \times \frac{7}{50}}{\frac{1}{5}} \\ &= \frac{3}{5}\end{aligned}$$

# Sally Clark case

Convicted of killing her two sons, on the basis of misuse of statistics.

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## Probability of Two Cot Deaths in Family

- For a family like Sally Clark's (non-smoking and affluent), the chances of a single cot death are 1 in 8543.
- Hence the chances of two cot deaths is 1 in $(8543 \times 8543)$, about 1 in 73 million.
- This is exceptionally unlikely, and hence two apparent cot deaths are strong evidence for murder.

## What is wrong with this logic?

# The Prosecutor's Fallacy

$$\mathbb{P}(Innocent|Evidence) \neq \mathbb{P}(Evidence|Innocent)$$

Bayes' Theorem tells us that:

$$\mathbb{P}(Innocent|E) = \frac{\mathbb{P}(E|Innocent)\mathbb{P}(Innocent)}{\mathbb{P}(E)}$$

and so:

$$\mathbb{P}(Innocent|E) = \frac{\mathbb{P}(E|Innocent)\mathbb{P}(Innocent)}{\mathbb{P}(E|Innocent)\mathbb{P}(Innocent) + \mathbb{P}(E|Guilty)\mathbb{P}(Guilty)}$$

# Medical Testing

A disease affects $1$ person in every $10,000$. There is a test for this disease. If the subject has the disease, the test comes back positive $98\%$of the time. If the subject does not have the disease, the test comes back negative $98\%$ of the time. Your patient has tested positive. What is the probability that they have the disease?

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Write $\mathcal{D}$ for the event of having the disease, and write $\mathcal{T}$ for the event of a postive test.

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- $\mathbb{P}(\mathcal{D}) = 0.0001$
- $\mathbb{P}(not\mathcal{D}) = 0.9999$
- $\mathbb{P}(\mathcal{T}|\mathcal{D}) = 0.98$
- $\mathbb{P}(\mathcal{T}|not\mathcal{D}) = 0.01$

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- $\mathbb{P}(\mathcal{T}|\mathcal{D}) = 0.98$
- $\mathbb{P}(\mathcal{T}|not\mathcal{D}) = 0.01$

Thus:

$$\mathbb{P}(\mathcal{D}|\mathcal{T}) = \frac{\mathbb{P}(\mathcal{T}|\mathcal{D})\mathbb{P}(\mathcal{D})}{\mathbb{P}(\mathcal{T})} = \frac{0.98 \times 0.0001}{\mathbb{P}(\mathcal{T})}$$

$$\mathbb{P}(not\mathcal{D}|\mathcal{T}) = \frac{\mathbb{P}(\mathcal{T}|not\mathcal{D})\mathbb{P}(not\mathcal{D})}{\mathbb{P}(\mathcal{T})} = \frac{0.01 \times 0.9999}{\mathbb{P}(\mathcal{T})}$$

But, we must have $\mathbb{P}(\mathcal{D}|\mathcal{T}) + \mathbb{P}(not\mathcal{D}|\mathcal{T}) = 1$ Hence:

$$\mathbb{P}(\mathcal{D}|\mathcal{T}) = \frac{(0.98 \times 0.0001)}{(0.98 \times 0.0001) + (0.01 \times 0.9999)} = 0.0097$$

# NOTE

If $A$ can only take a finite number of forms, $A_1, ..., A_n$, then, since $\sum_{i=1}^{n} \mathbb{P}(A_i|B) = 1$:

$$\mathbb{P}(A_i|B) = \frac{\mathbb{P}(B|A_i)\mathbb{P}(A_i)}{\sum_{j=1}^{n} \mathbb{P}(B|A_j)\mathbb{P}(A_j)}$$