Bayes' Theorem and Conditional Probability in the Real World

Solution Sheet

Mary Fortune
Liverpool Maths Club - 20 September 2014

1. *There are* 11824 *undergraduates at the University of Cambridge:* 671 *of them are from Trinity College, and* 588 *are from St John's College. All Trinity members would rather go to Oxford than St John's, and all St John's members would rather go to St John's than Oxford. Students from all other colleges are equally likely to prefer Oxford to St John's. You ask a randomly selected student, who informs you that they would rather go to Oxford than St John's. What is the probability that they are Trinitarian?* Write $\mathcal{T}$ for a Trinitarian, and $\mathcal{R}$ for someone who would rather go to Oxford than St John's.

$$\begin{aligned} \mathbb{P}(\mathcal{T}|\mathcal{R}) &= \frac{\mathbb{P}(\mathcal{R}|\mathcal{T})\mathbb{P}(\mathcal{T})}{\mathbb{P}(\mathcal{R})} \\ &= \frac{1 \times \frac{671}{11824}}{\left(\frac{671+\frac{11824-671-588}{2}}{11824}\right)} \\ &\sim 0.1127 \end{aligned}$$

2. *A town has two taxi companies: Blue Birds, whose cabs are blue, and Night Owls, whose cabs are black. Blue Birds has 15 taxis in its fleet, and Night Owls has 75. Late one night, there is a hit-and-run accident involving a taxi. The town's taxis were all on the streets at the time of the accident. A witness saw the accident and claims that a blue taxi was involved. At the request of the police, the witness undergoes a vision test under conditions similar to those on the night in question. Presented repeatedly with a blue taxi and a black taxi, in random order, he shows he can successfully identify the color of the taxi 4 times out of 5. Which company is more likely to have been involved in the accident?*

$$\begin{aligned} \mathbb{P}(Blue|Witness) &= \frac{\mathbb{P}(Witness|Blue)\Pi(Blue)}{\mathbb{P}(Witness|Blue)\Pi(Blue)+\mathbb{P}(Witness|Black)\Pi(Black)} \\ &= \frac{0.8 * \frac{15}{90}}{0.8 * \frac{15}{90} + 0.2 * \frac{75}{90}} \\ &= \frac{4}{9} \end{aligned}$$

3. *Bruce Banner is awaiting cosmetic surgery which will change his appearance again (the third time he will have done so), so that authorities will not recognise him as the alter ego of the Incredible Hulk. There is a* 1% *chance that the surgeon is a supervillain who will recognise Banner and take the opportunity to inject him with a slow-acting disease which will kill the Banner with probability* $\frac{1}{2}$ *for each day that he remains alive. (So if Banner survives Day One with probability* $\frac{1}{2}$ *he will die with probability* $\frac{1}{2}$ *on Day Two.) Nine days after the surgery, Banner (in his new identity) is still alive and well and undetected*

as the Hulk. What is the probability that the Hulk will still be in this state at the end of Day Ten?

Write $\mathcal{P}$ for the event that Banner has been poisoned, $\mathcal{S}_9$ for the event that Banner survives until Day 9, and $\mathcal{S}_{10}$ for the event that Banner survives until Day 10. Then:

$$\begin{aligned}
\mathbb{P}(\mathcal{S}_{10}|\mathcal{S}_9) &= \mathbb{P}(\mathcal{S}_{10}|\mathcal{S}_9,\mathcal{P}) \times \mathbb{P}(\mathcal{P}|\mathcal{S}_9) + \mathbb{P}(\mathcal{S}_{10}|\mathcal{S}_9, not\mathcal{P}) \times \mathbb{P}(not\mathcal{P}|\mathcal{S}_9) \\
&= \frac{1}{2}\mathbb{P}(\mathcal{P}|\mathcal{S}_9) + \mathbb{P}(not\mathcal{P}|\mathcal{S}_9) \\
&= 1 - \frac{1}{2}\mathbb{P}(\mathcal{P}|\mathcal{S}_9) \\
&= 1 - \frac{1}{2}\left(\frac{\mathbb{P}(\mathcal{S}_9|\mathcal{P})\mathbb{P}(\mathcal{P})}{\mathbb{P}(\mathcal{S}_9|\mathcal{P})\mathbb{P}(\mathcal{P}) + \mathbb{P}(\mathcal{S}_9|not\mathcal{P})\mathbb{P}(not\mathcal{P})}\right) \\
&= 1 - \frac{1}{2}\left(\frac{\frac{1}{2^9}\frac{1}{100}}{\frac{1}{2^9}\frac{1}{100} + \frac{99}{100}}\right) \\
&= \frac{101377}{101378}
\end{aligned}$$

4. *You are in Las Vegas. Late one night in a bar you meet a guy who claims to know that in the casino at the Tropicana there are two sorts of slot machines: one that pays out 10% of the time, and one that pays out 20% of the time [note these numbers may not be very realistic]. The two types of machines are coloured red and blue. The only problem is, the guy is so drunk he cant quite remember which colour corresponds to which kind of machine. Next day you go to the Tropicana to find out more. You find a red and a blue machine side by side. You toss a coin to decide which machine to try first; based on this you then put the coin into the red machine.*

   (a) *It doesnt pay out. How should you update your estimate of the probability that this is the machine youre interested in?*

   (b) *What if it had paid out - what would be your new estimate then?*

Write $\mathcal{R}$ for the event that the red machine pays out more, write $\mathcal{B}$ for the event that the blue machine pays out more, and $\mathcal{P}$ for the event that the machine pays out.

(a)

$$\begin{aligned}
\mathbb{P}(\mathcal{R}|not\mathcal{P}) &= \frac{\mathbb{P}(not\mathcal{P}|\mathcal{R})\mathbb{P}(not\mathcal{P})}{\mathbb{P}(not\mathcal{P}|\mathcal{R})\mathbb{P}(not\mathcal{P}) + \mathbb{P}(not\mathcal{P}|\mathcal{B})\mathbb{P}(not\mathcal{B})} \\
&= \frac{0.8 \times 0.5}{0.8 \times 0.5 + 0.9 \times 0.5} \\
&= \frac{8}{17}
\end{aligned}$$

(b)

$$\begin{aligned}
\mathbb{P}(\mathcal{R}|\mathcal{P}) &= \frac{\mathbb{P}(\mathcal{P}|\mathcal{R})\mathbb{P}(\mathcal{P})}{\mathbb{P}(\mathcal{P}|\mathcal{R})\mathbb{P}(\mathcal{P}) + \mathbb{P}(\mathcal{P}|\mathcal{B})\mathbb{P}(\mathcal{B})} \\
&= \frac{0.2 \times 0.5}{0.2 \times 0.5 + 0.1 \times 0.5} \\
&= \frac{2}{3}
\end{aligned}$$

5. *Recall the medical testing problem from earlier. A disease affects* 1 *person in every* 10,000*. There is a test for this disease. If the subject has the disease, the test comes back positive* 98%*of the time. If the subject does not have the disease, the test comes back negative* 98% *of the time.*

   (a) *Your patient tests positive on 2 independent tests. What is the probability that they have the disease?*

   (b) *Your patient tests positive on n independent tests. What is the probability that they have the disease?*

$$\mathbb{P}(\mathcal{D}|\mathcal{T}) = \frac{(\mathbb{P}(\mathcal{T}|\mathcal{D}))^n \mathbb{P}(\mathcal{D})}{\mathbb{P}(\mathcal{T})}$$

$$\mathbb{P}(not\mathcal{D}|\mathcal{T}) = \frac{(\mathbb{P}(\mathcal{T}|not\mathcal{D}))^n \mathbb{P}(not\mathcal{D})}{\mathbb{P}(\mathcal{T})}$$

Hence:

$$\mathbb{P}(\mathcal{D}|\mathcal{T}) = \frac{(0.98^n \times 0.0001)}{(0.98^n \times 0.0001) + (0.01^n \times 0.9999)}$$

6. *Nicole Brown was murdered at her home in Los Angeles on the night of June 12,1994. The Prime suspect was her husband 0.J.Simpson, at the time a well-known celebrity famous both as a TV actor and as a retired professional football star. This murder led to one of the most heavily publicized murder trial in U.S. during the last century. The fact that the murder suspect had previously physically abused his wife played an important role in the trial. The famous defense lawyer Alan Dershowitz, a member of the team of lawyers defending the accused, tried to belittle the relevence of the fact by stating that only* 0.1% *of the men who physically abuse their wives actually end up murdering them. Was the fact that O.J.Simpson had previously physically abused his wife irrelevant to the case?*

   In this particular case it is important to condition on the crucial fact that Nicole Brown was murdered. It is wrong to estimate the probability uniformly on possible outcomes. The question, therefore, is not what the probability is that abuse leads to murder, but the probability that the husband is guilty in light of the fact that he had previously abused his wife.

7. *Let p be the probability of getting a Head on a toin coss.*

   (a) *You know that p is in the set $\{0, \frac{1}{10}, \frac{2}{10}, ..., \frac{9}{10}, 1\}$. Initially, you believe that each of these is equally likely. On your first toss, you get a Head. Now, what do you believe about the distribution of p?*

   (b) ***For those who know about continuous probability distributions**. You know that p could have any value in the set $[0, 1]$, and believe that it has an uniform distribution. On your first toss, you get a Head. Now, what do you believe about the distribution of p?*

   (a)

$$\begin{aligned} \mathbb{P}(p = \frac{i}{10}|Head) &= \frac{\mathbb{P}(Head|p = \frac{i}{10})\Pi(p = \frac{i}{10})}{\sum_{i=0}^{10} \mathbb{P}(Head|p = \frac{i}{10})\Pi(p = \frac{i}{10})} \\ &= \frac{\frac{i}{10}}{\sum_{i=0}^{10} \frac{i}{10}} \\ &= \frac{i}{55} \end{aligned}$$

(b)

$$\begin{aligned} f(p|Head) &= \frac{\mathbb{P}(Head|p)\Pi(p)}{\int_0^1 \mathbb{P}(Head|x)\Pi(x)dx} \\ &= \frac{p}{1/2} \\ &= 2p \end{aligned}$$