Recurrence Relations: Answers

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Liverpool Maths Club, March 2013

Problem 1

A)

$$2, 8, 32, 128, 512, \ldots$$

i) it's clear that every term is 4 times more than its predecessor so

$$u_n = 4u_{n-1}$$

Note: $u_{n+1} = 4u_n$ is also perfectly fine and means the same thing.

ii) $u_n = 2 \times 4^n$

B) Looking at the first few terms:

$$u_0 = a$$

$$u_1 = a \times r$$

$$u_2 = a \times r \times r = ar^2$$

$$u_3 = a \times r \times r \times r = ar^3$$

etc...

so we can see that

$$u_n = ar^n$$

This type of sequence is called a *geometric sequence* and the formula stated above gives the general term for any sequence of this type.

Problem 2

A) The Fibonacci Sequence is defined by $F_n = F_{n-1} + F_{n-2}$ with $F_0 = 0$ $F_1 = 1$

i) Trying solutions of the form $F_n = a^n$ we have

$$a^n = a^{n-1} + a^{n-2}$$

rearranging and dividing by a^{n-2} gives us the auxiliary equation

$$a^2 - a - 1 = 0$$

this is solved for a using the quadratic formula and we get

$$a = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-1)}}{2} = \frac{1 \pm \sqrt{5}}{2}$$

so solutions for F_n are of form

$$F_n = A\left(\frac{1+\sqrt{5}}{2}\right)^n + B\left(\frac{1-\sqrt{5}}{2}\right)^n$$

Now we were given that $F_0 = 0$ and $F_1 = 1$ so

$$0 = A\left(\frac{1+\sqrt{5}}{2}\right)^0 + B\left(\frac{1-\sqrt{5}}{2}\right)^0 = A + B \tag{1}$$

and

$$1 = A\left(\frac{1+\sqrt{5}}{2}\right) + B\left(\frac{1-\sqrt{5}}{2}\right) \tag{2}$$

$$A + B = 0 \implies A = -B$$

substituting this into (2)

$$1 = A\left(\frac{1+\sqrt{5}}{2}\right) - A\left(\frac{1-\sqrt{5}}{2}\right) = A\left(\frac{1+\sqrt{5}}{2} - \frac{1-\sqrt{5}}{2}\right)$$

$$\implies A = \frac{1}{\sqrt{5}}$$

$$B = -\frac{1}{\sqrt{5}}$$

so

$$F_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right)$$

ii) Substituting the recurrence relation for the fibonacci sequence in place of F_n we get

$$\frac{F_n}{F_{n-1}} = \frac{F_{n-1}+F_{n-2}}{F_{n-1}} = \frac{F_{n-1}}{F_{n-1}} + \frac{F_{n-2}}{F_{n-1}} = 1 + \frac{F_{n-2}}{F_{n-1}} = 1 + \frac{1}{\frac{F_{n-1}}{F_{n-2}}}$$

Now $\frac{F_{n-1}}{F_{n-2}} \to \frac{F_n}{F_{n-1}}$ as $n \to \infty$ so at $n = \infty$ we can write $\frac{F_{n-1}}{F_{n-2}} = \frac{F_n}{F_{n-1}} = L$
and so

$$\frac{F_n}{F_{n-1}} = 1 + \frac{1}{\frac{F_{n-2}}{F_{n-1}}}$$

transforms to

$$L = 1 + \frac{1}{L}$$

multiplying through by L and rearranging gives

$$L^2 - L - 1 = 0$$

which when using the quadratic formula we find

$$L = \frac{1 \pm \sqrt{5}}{2}$$

It is clear that L must be positive so

$$\lim_{n\to\infty} \frac{F_n}{F_{n-1}} = \frac{1+\sqrt{5}}{2} = 1.618\ldots$$

You might recognise this as the golden ratio (normally denoted φ).

iii) This means that for large n

$$F_n \approx \varphi F_{n-1}$$

Given $F_{10} = 55$, this approximation suggests

$$F_{11} \approx \varphi F_{10} \approx 1.618 \times 55 = 88.99$$

so this shows that

$$F_{11} \approx 88.99$$

so

$$F_{11} = 89$$

checking this by using the recurrence relation to find F_{11} we can write down terms of the sequence up to F_{11}:

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89$$

and so indeed $F_{11} = 89$ and our approximation gives an accurate result.

iv) From the answer to part (ii) we saw that

$$\lim_{n\to\infty} \frac{F_n}{F_{n-1}} = \varphi$$

$$\frac{F_n}{F_{n-1}} = \frac{F_{n-1} + F_{n-2}}{F_{n-1}} = 1 + \frac{F_{n-2}}{F_{n-1}} = 1 + \cfrac{1}{\cfrac{F_{n-1}}{F_{n-2}}}$$

$$= 1 + \cfrac{1}{1 + \cfrac{F_{n-3}}{F_{n-2}}} = 1 + \cfrac{1}{1 + \cfrac{1}{\cfrac{F_{n-2}}{F_{n-3}}}} = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{\cfrac{F_{n-3}}{F_{n-4}}}}} = \dots$$

if n is infinitely large, this process can be repeated infinitely many times to get

$$\lim_{n\to\infty}\frac{F_n}{F_{n-1}} = \varphi = 1 + \cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{\ddots}}}}}$$

This is the simplest form of never-ending continued fraction.

B)

i) Counting the numbers of ways from one up to 5 stairs the answers are 1,2,3,5,8. It can be seen that every term is a sum of the previous two. Let's call the number of ways we can climb n steps u_n the recurrence relation is

$$u_n = u_{n-1} + u_{n-2}$$

Does this work in general?
Well if we think about how we can get to the last step, we can either climb one step rom the stair immediately below, or climb two from the stair two below. So the number of ways we can get to the top step is a sum of the number of ways we can get to the previous two, so indeed the recurrence relation for the whole sequence is

$$u_n = u_{n-1} + u_{n-2}$$

ii) This is the same recurrence relation as in part **A(i)** so to help us out we can use the answer we got which was

$$F_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right)$$

BUT we were given that $u_1 = 1$ and $u_2 = 2$ which are actually equal to F_0 and F_1 respectively. To allow for this we must alter the formula slightly, replacing n with $n+1$ so our formula for the number of ways (u_n) we can climb n stairs is

$$u_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right)$$

Problem 3

This is quite a tricky question. If you find the answer to this problem, it would be nice if you could email me at yyanis@me.com with your explanation!