Liverpool University Maths Club

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Sums of Squares

Peter Giblin

Can you write 35 as a sum of three squares? (Only whole numbers are to be used, so the squares are 1, 4, 9, 16, 25......)

$35 = 5^2 + 3^2 + 1^2$

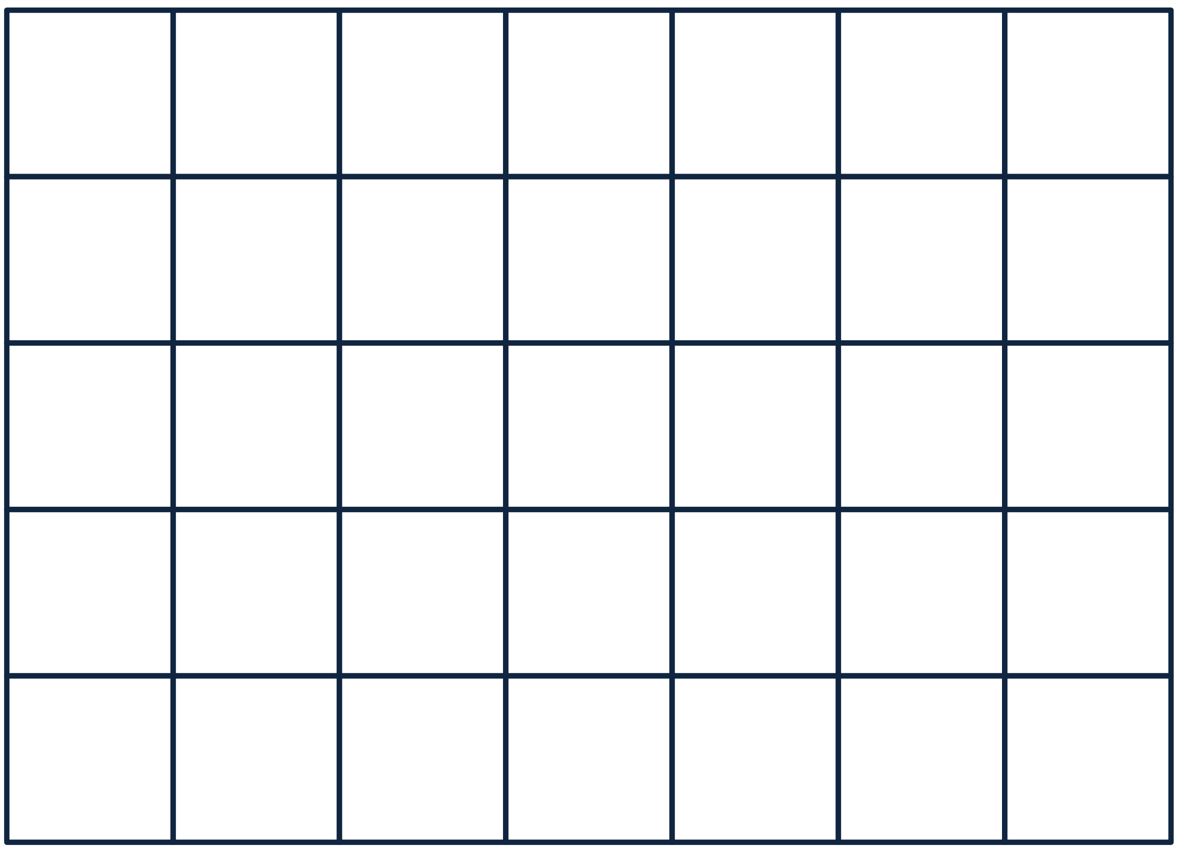
What about as a sum of four squares? (apart from adding 0^2 to the previous solution!)

$35 = 4^2 + 3^2 + 3^2 + 1^2$

Actually these are the only solutions.

One way of writing 35 as a sum of five squares is

$35 = 5^2 + 2^2 + 2^2 + 1^2 + 1^2$



Now can you fit squares of sides
5, 2, 2, 1, 1
inside this
5 x 7
rectangle to cover it exactly?

$35 = 5^2 + 3^2 + 1^2$

$35 = 4^2 + 3^2 + 3^2 + 1^2$

Can these expressions be used to fit three squares, or four squares, inside a 5 x 7 rectangle to exactly cover it?

No, so what is the smallest number of squares which can be fitted together to exactly cover a 5 x 7 rectangle?

five squares

There are other ways of covering a 5 x 7 rectangle with squares, e.g. can you fit together squares of sides

3, 3, 2, 2, 2, 2, 1

to exactly cover a 5 x 7 rectangle?

Incidentally can 35 be written as a sum of **two** squares?

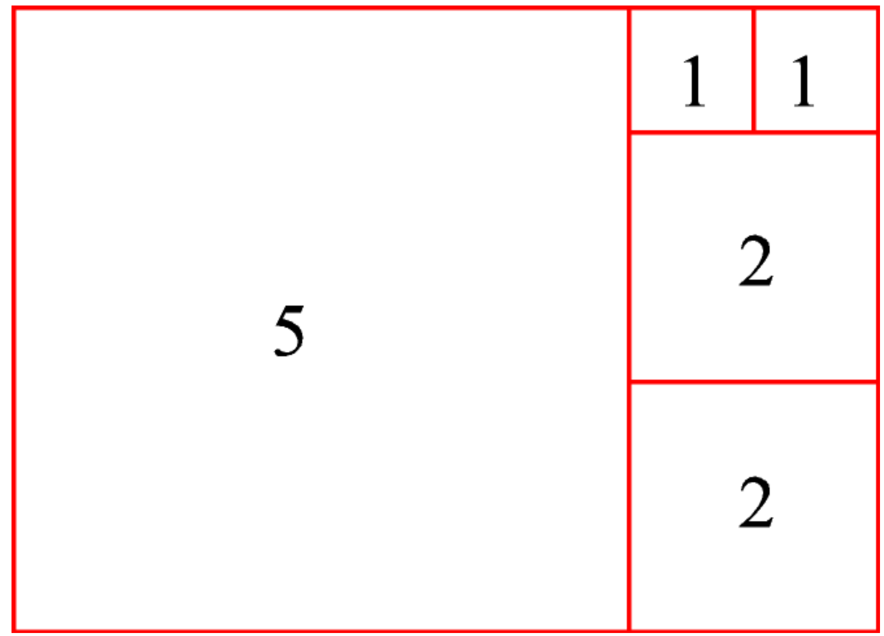
Here’s a remarkable general fact: **No number which gives remainder 3 when divided by 4 can be written as the sum of two squares.**

Squaring an even number gives always remainder 0 when divided by 4

Squaring an odd number always gives remainder 1 when divided by 4

So adding **two** of these always gives remainder 0, 1 or 2.....never remainder 3

Here’s a general method for dividing any rectangle into squares:



Take say 5 x 7

$7 = \underline{1} \times 5 + 2$

$5 = \underline{2} \times 2 + 1$

$2 = \underline{2} \times 1 + 0$

Now use this as a recipe for fitting squares together. The number of squares used is the sum of the numbers which are underlined.

Another example, 5 x 6

6 = 1 x 5 + 1
5 = 5 x 1 + 0

This gives a covering of the 5 x 6 rectangle with 1 + 5 = 6 squares.

Can it be done with fewer than six squares?

The method of successive division is called the **euclidean algorithm** and in fact the last nonzero remainder is always the greatest common divisor of the numbers you start with.

Another example: 5 x 8

How many squares does the euclidean algorithm method give?

It gives five squares and as it happens this is the fewest possible with a 5 x 8 rectangle.

(The only way of writing 40 as a sum of 4 squares is $4^2 + 4^2 + 2^2 + 2^2$ which can't be used.)

Notice that the method gives
$5 \times 8 = 5^2 + 3^2 + 2^2 + 1^2 + 1^2$

and remember the Fibonacci sequence

1, 1, 2, 3, 5, 8, 13, 21, ... where you add two numbers to get the next.

In fact the same construction shows, in general, that

the sum of the squares of the first n terms of the Fibonacci sequence equals the product of the n^{th} and $(n+1)^{st}$ terms

For example an 8 x 13 rectangle would give

$13 = \underline{1} \times 8 + 5$
$8 = \underline{1} \times 5 + 3$
$5 = \underline{1} \times 3 + 2$
$3 = \underline{1} \times 2 + 1$
$2 = \underline{2} \times 1 + 0$

so six squares with sides 8, 5, 3, 2, 1, 1 fit together to make a 8 x 13 rectangle.

Incidentally there are other more challenging problems with fitting squares to make rectangles....

A 11 x 15 rectangle can be covered with seven squares using the euclidean algorithm method....

But it is also possible to use nine squares and have an arrangement which contains **no smaller rectangles** (a so-called ‘simple dissection of the rectangle into squares’). The nine squares have sides 6,6,5,5,4,4,3,1,1.

But I now want to look at the euclidean algorithm a different way.

$$\frac{9}{5} = 1 + \frac{4}{5}$$

9 = 1 x 5 + 4: divide through by 5

5 = 1 x 4 + 1: divide through by 4

4 = 4 x 1 + 0: divide through by 1

$$\frac{5}{4} = 1 + \frac{1}{4}$$

so

$$\frac{4}{5} = \frac{1}{1 + \frac{1}{4}}$$

and

write this **continued fraction** as [1,1,4]

$$\frac{9}{5} = 1 + \frac{1}{1 + \frac{1}{4}}$$

Similarly $\frac{20}{7}$ = [2,1,6]

$\frac{8}{5}$ = [1,1,1,2]

But let's consider [1,2,2,2,2,.....]

$$x = 1 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2 + \ldots}}}$$

We can use this to find x

$$x = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \dots}}}$$

This is 1 + x

So $x = 1 + \frac{1}{1+x}$ which gives

$x - 1 = \frac{1}{1+x}$

$x^2 - 1 = 1, \quad x^2 = 2$

so $x = \pm\sqrt{2}$, but clearly $x > 0$

so $x = \sqrt{2}$.

We can also consider this as the limit of the ‘convergents’ of the continued fraction

[1,2] = 1.5
[1,2,2] = 1.4
[1,2,2,2] = 1.41666...
[1,2,2,2,2] = 1.413793103...
[1,2,2,2,2,2] = 1.414201184....

where in each case if x is one of these then the next one is $1 + 1/(1+x)$

These are convergents with limit $\sqrt{2}$

Extra examples:

What about

[1,1,1,1,1,1.....]

[1,1,2,1,2,1,2,1,2,1,2,.....]

[d,2d,2d,2d,2d,2d,2d,....] = $\sqrt{(d^2 + 1)}$, $d \geq 1$

[d-1,1,2d-2,1,2d-2,1,2d-2,....] = $\sqrt{(d^2 - 1)}$, $d \geq 2$

In fact the mathematical constant e is given by a regular continued fraction

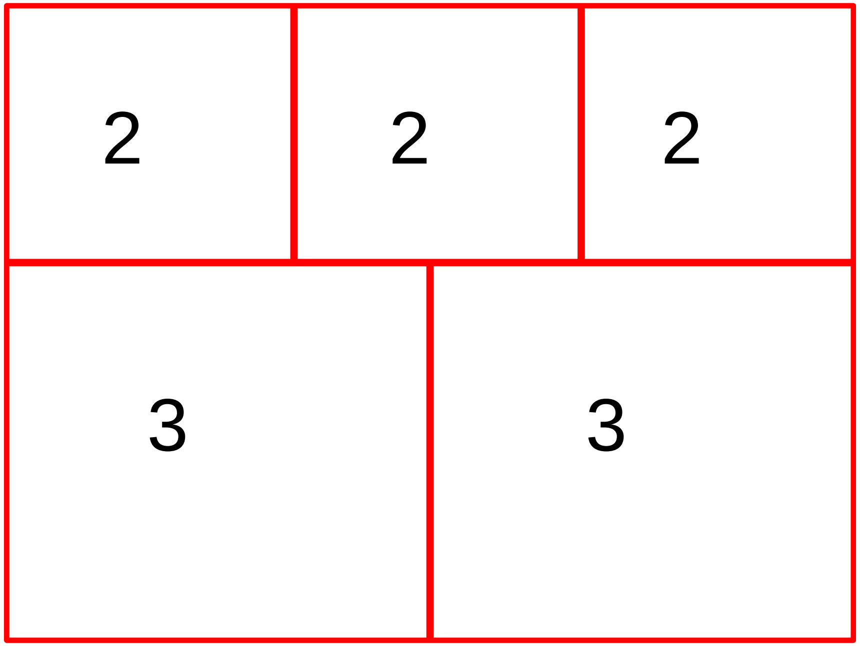
e = [1,0,1,1,2,1,1,4,1,1,6,1,1,8,....]

However π as a continued fraction has a structure which is **unknown**, that is there is no known regularity in the numbers occurring even though it has been calculated to 6,000,000,000 terms!

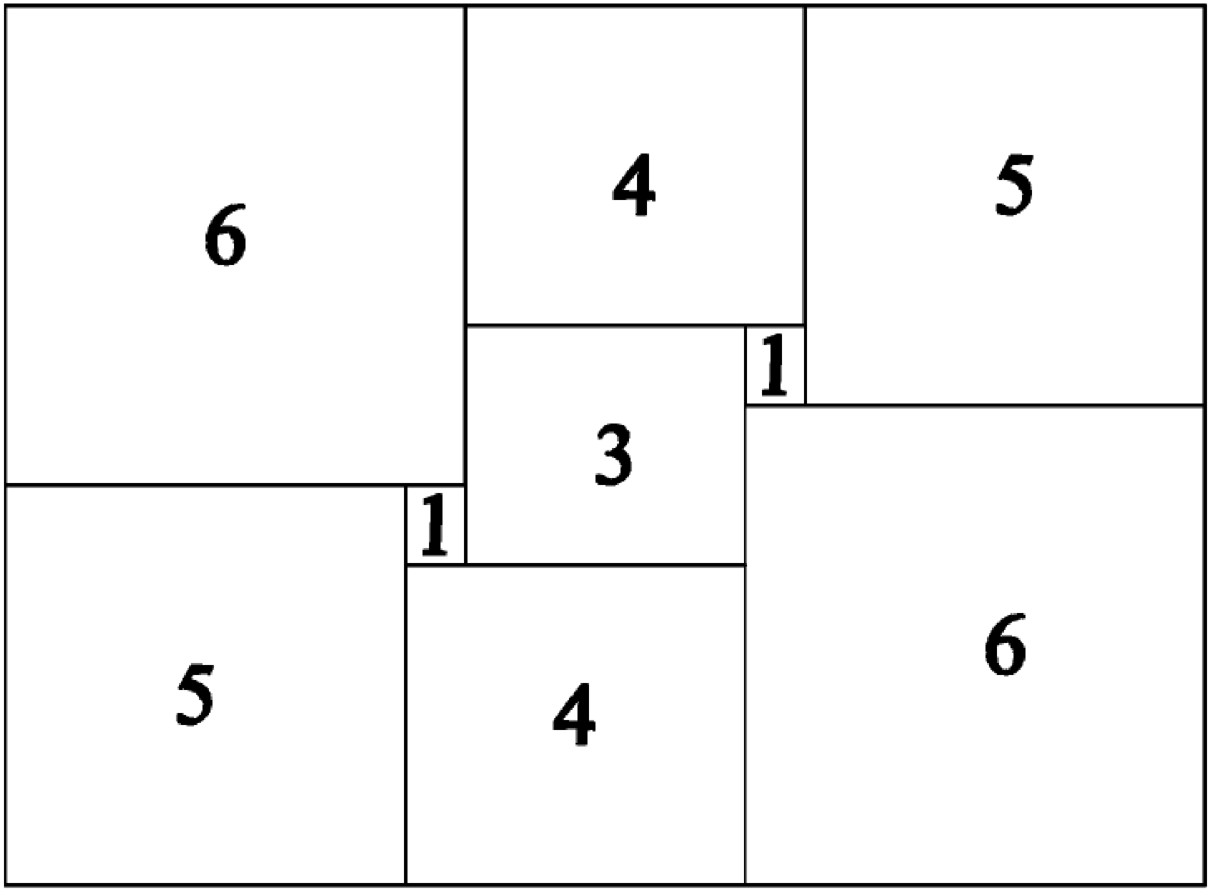
π = [3,7,15,1,292,1,1,1,2,1,3,1,....]

Note that [3,7] = 22/7
[3,7,15] = 333/106,
[3,7,15,1] = 355/113 = 3.14159292....

Answer to 5 x 6 rectangle using five squares:



What about more general rectangles, $n \times (n - 1)$, where n is even?



A ‘simple’ dissection of an 11 x 15 rectangle into squares (no subset of squares makes a rectangle)

Some web references

http://en.wikipedia.org/wiki/Fermat%27s_theorem_on_sums_of_two_squares

http://en.wikipedia.org/wiki/Lagrange%27s_four-square_theorem

http://mathworld.wolfram.com/PerfectSquareDissection.html

http://en.wikipedia.org/wiki/Squaring_the_square

http://www.blackdouglas.com.au/taskcentre/138recsq.htm

also web references to ‘Mrs Perkins’s Quilt’ such as

http://www.maa.org/editorial/mathgames/mathgames_12_01_03.html

http://ime.math.arizona.edu/ati/Math%20Projects/C1_MathFinal_Papenfus.pdf