Multiplication Tables

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	1	2	3	4
1	1	2	3	4
2	2	4	6	8
3	3	6	9	12
4	4	8	12	16

An ordinary multiplication table

But I’m interested in ones that look more like this:

	1	2	3	4
1	1	2	3	4
2	2	4	1	3
3	3	1	4	2
4	4	3	2	1

Can you see/guess how this is constructed?

or this?

	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

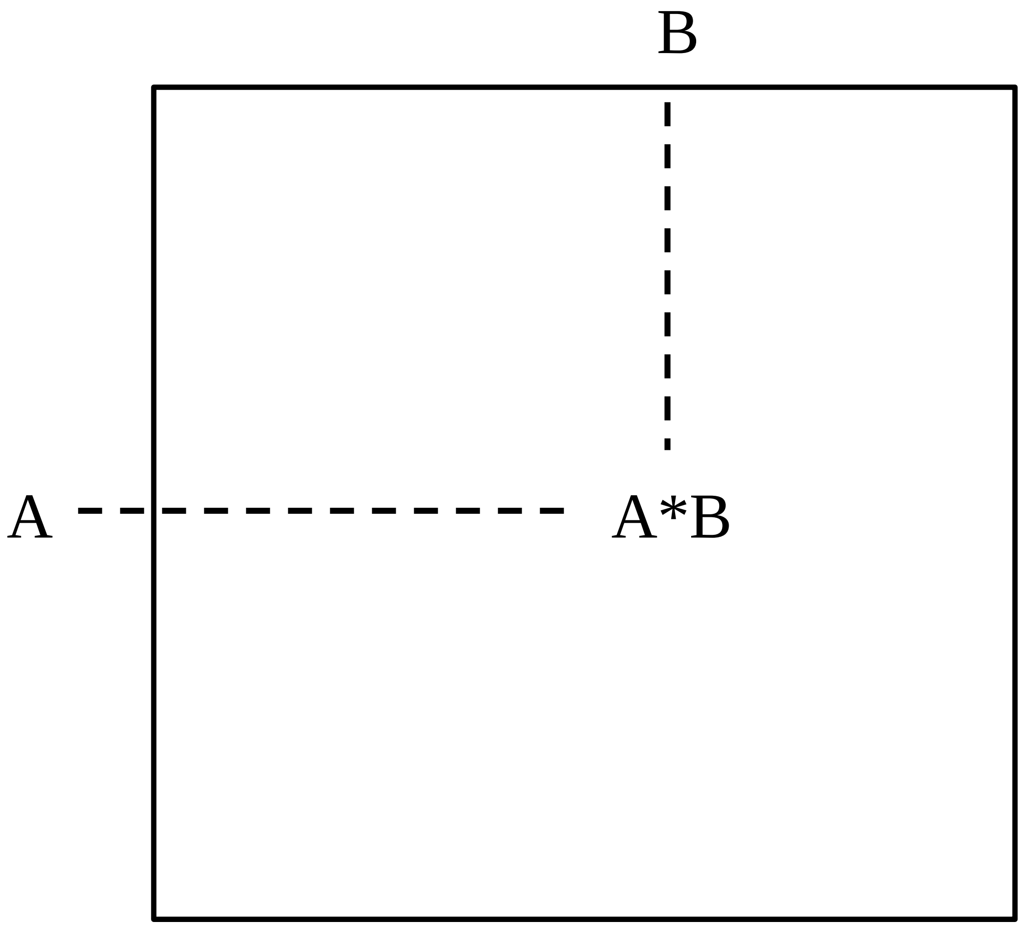
which is really more like **addition** than multiplication

So we'd better just call these **tables.** (They are also called **Latin squares.)** The **order** is the size of the square, so the above is of order 4.

	1	2	3	4
1	1	2	3	4
2	2	4	1	3
3	3	1	4	2
4	4	3	2	1

In the last two examples the entries in the table are the same as the numbers being 'multiplied' or 'combined'

We might write A*B for the entry given by row A and column B:



so * might stand for various things involving multiplication, addition etc. but always the same thing throughout a given table

We shall only consider tables (order *n*) where each row and each column contains the same *n* symbols.

So for the tables below can you find what the rule is?

	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

	0	1	2
0	0	2	1
1	2	1	0
2	1	0	2

All the tables so far are **commutative**
(meaning A*B = B*A always)

The second table above is **idempotent**
(meaning A*A = A always)

Is there a commutative, idempotent table of order 4?

	A	B	C	D
A	A			
B		B		
C			C	
D				D

We want this to be symmetric about the diagonal to make it commutative. And as always every row and every column must contain the letters A, B, C, D in some order

Hint: consider the number of A’s which appear in the table altogether.

For the case of order 5, try using 0, 1, 2, 3, 4 and the formula

A*B = 3(A+B)   mod 5
(i.e. remainder of 3(A+B) after division by 5)

In a sense A*B is the **average** of A and B mod 5 since in fact 2(A*B) = A+B, just like for ordinary averages.

Maybe you can see how a similar trick might work for order 7, producing a commutative, idempotent table.

Orthogonal Tables
(Orthogonal Latin Squares)

Consider these two tables of order 3 (I'll just write the nine entries) and then write the pairs of entries:

0	1	2
1	2	0
2	0	1

0	1	2
2	0	1
1	2	0

00	11	22
12	20	01
21	02	10

Each of the nine pairs from 0,1,2 appears just once

Can you find an orthogonal table to this one?

0	1	2	3
1	0	3	2
2	3	0	1
3	2	1	0

It can start like this:

0	1	2	3

so the pairs start like this

00	11	22	33
1	0	3	2
2	3	0	1
3	2	1	0

Thus the four places where 0 occurs in the first table must be occupied by different numbers in the second table, and similarly for 1,2,3.

In fact there are two solutions, and any pair among these three are orthogonal:

0	1	2	3
1	0	3	2
2	3	0	1
3	2	1	0

0	1	2	3
3	2	1	0
1	0	3	2
2	3	0	1

0	1	2	3
2	3	0	1
3	2	1	0
1	0	3	2

Orthogonal tables have many applications in the design of experiments but here is a simple application: place the A, J, Q, K of the four suits in a 4 x 4 square so that every row, every column and the two diagonals contain all four suits.

A♠	K♥	Q♦	J♣
Q♣	J♦	A♥	K♠
J♥	Q♠	K♣	A♦
K♦	A♣	J♠	Q♥

is one solution, using the second and third tables above

For order 5 there is an amazing method of producing FOUR mutually orthogonal squares. Start with

	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

which is addition modulo 5:
A*B = A+B mod 5

Now write down the table
A*B = A+2B mod 5
Check that it is orthogonal to the table on the left.

Maybe you can find how to check this without actually working out the second table!

In fact the tables
A+B, A+2B, A+3B, A+4B
are mutually orthogonal.

Here is an illustration of this which comes from Wikipedia (article on orthogonal latin squares)

fjords	jawbox	phlegm	qiviut	zincky
zincky	fjords	jawbox	phlegm	qiviut
qiviut	zincky	fjords	jawbox	phlegm
phlegm	qiviut	zincky	fjords	jawbox
jawbox	phlegm	qiviut	zincky	fjords

There is even an order 5 table which is orthogonal to its transpose (transpose = reflection in the main diagonal):

A	B	C	D	E
C	E	D	B	A
D	A	B	E	C
E	D	A	C	B
B	C	E	A	D

using A B C D E for the entries for a change

It was proved in 1974 that tables orthogonal to their transposes exist for all orders except 2, 3 and 6.

Maybe you can find an order 4 example.

Key words to look up more information:

Latin squares

Graeco-Latin squares

Orthogonal Latin squares

Block designs

Euler’s work on magic squares and Latin squares