Synthematic Totals, by Ian Porteous

Maths Club, 29 January, 2011.

4.

Given the four numbers 1, 2, 3, 4 there are (4x3)/2 = 6 *duads* (12), (13), (14), (23), (24), (34) and 6/2 = 3 duad pairs (12)(34), (13)(24), (14)(23). Now there are 24 permutations of 1, 2, 3, 4 and each of these will induce a permutation of the three duad pairs, there being 6 such. Four of the permutations leave the three alone, namely the identity, which does nothing, swapping 1 and 2 and at the same time swapping 3 and 4, swapping 1 and 3 and at the same time swapping 2 and 4 and swapping 1 and 4 and at the same time swapping 2 and 3. It is no accident that 24/4 = 6.

6.

Given the six numbers 1, 2, 3, 4, 5, 6 there are (6x5)/2 = 15 duads (12) etc., and (15x6 /6 = 15 duad triples known as *synthemes* (12)(34)(56), etc.. A *total* is a set of 15 synthemes, involving each of the 15 duads once and once only. What we find is that there are exactly 6 totals, any particular syntheme belonging to exactly two totals. We call the totals A, B, C, D, E, F in some order. The first table shows them all. The remarkable thing is that we can go back from these six letters to the original six numbers by repeating the process that enabled us to find them. A duad of totals determines the unique syntheme belonging to each of them, for example (AB) determines the syntheme (12)(34)(56), while a syntheme of totals determines a duad, for example (AB)(CD)(EF) determines the three synthemes (12)(34)(56), (12)(35)(46) and (12)(36)(45) that have the duad (12) in common. Finally, a total of totals determines five duads all of which have just one of 1, 2, 3, 4, 5, 6 in common, that number then representing the total of totals.

Any permutation of the six numbers induces a permutation of the six letters, there being 6! = 720 of these, and the correspondence is invertible. All this occurs only for the number 6. There is nothing analogous for any other number.

Dice.

Each syntheme can be represented by exactly two dice, the elements of each of the three duads lying on opposite faces, one of the dice being the reflection of the other. For example, the standard die represents the syntheme (16)(25)(34). If we orient each duad, so that (21) = -(12), then the synthemes are orientable as well in the obvious way, and it is these oriented synthemes that are represented by the dice. The second table shows an array of all the possible 30 dice.

Table 1

Synthematic Totals

	A	B	C	D	E	F
A		(12)(34)(56)	(13)(26)(45)	(14)(25)(36)	(15)(23)(46)	(16)(24)(35)
B	(12)(34)(56)		(15)(24)(36)	(16)(23)(45)	(14)(26)(35)	(13)(25)(46)
C	(13)(26)(45)	(15)(24)(36)		(12)(35)(46)	(16)(25)(34)	(14)(23)(56)
D	(14)(25)(36)	(16)(23)(45)	(12)(35)(46)		(13)(24)(56)	(15)(26)(34)
E	(15)(23)(46)	(14)(26)(35)	(16)(25)(34)	(13)(24)(56)		(12)(36)(45)
F	(16)(24)(35)	(13)(25)(46)	(14)(23)(56)	(15)(26)(34)	(12)(36)(45)	

Table 2

Oriented Synthematic Totals

	A	B	C	D	E	F
A	0	-(12)(34)(56)	-(13)(26)(45)	-(14)(25)(36)	-(15)(23)(46)	-(16)(24)(35)
B	(12)(34)(56)	0	-(15)(24)(36)	-(16)(23)(45)	-(14)(26)(35)	-(13)(25)(46)
C	(13)(26)(45)	(15)(24)(36)	0	-(12)(35)(46)	-(16)(25)(34)	-(14)(23)(56)
D	(14)(25)(36)	(16)(23)(45)	(12)(35)(46)	0	-(13)(24)(56)	-(15)(26)(34)
E	(15)(23)(46)	(14)(26)(35)	(16)(25)(34)	(13)(24)(56)	0	-(12)(36)(45)
F	(16)(24)(35)	(13)(25)(46)	(14)(23)(56)	(15)(26)(34)	(12)(36)(45)	0

Liverpool University Maths Club 29 January 2011
Duads, Synthemes and Totals
Ian Porteous, notes by Peter Giblin

For internet references type `synthematic total` into Google.

Start with the numbers 1,2,3,4,5,6. A **duad** is a pair of these numbers, but the order does not matter. So (12) and (21) are the same duad.

1. How many duads are there? Answer: There are 6 choices for the first number and 5 for the second so this gives $6 \times 5 = 30$. But this counts each one twice (e.g. counts (12) and (21) separately) so the correct answer is $30/2 = \mathbf{15}$.

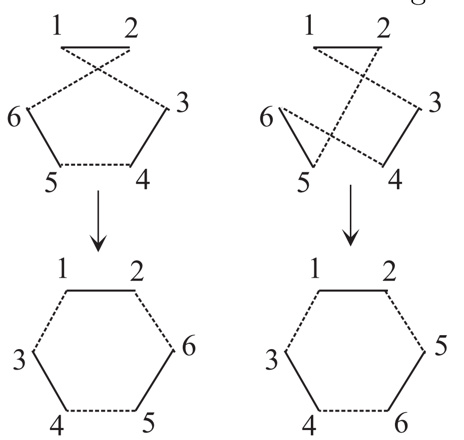
They are $(12), (13), (14), (15), (16), (23), (24), (25), (26), (34), (35), (36), (45), (46), (56)$.

A **syntheme** is a trio of duads which uses all the numbers 1,2,3,4,5,6. For example (12)(34)(56). However the order of the duads does not matter so this is the same syntheme as (34)(56)(12), or (34)(12)(56) etc.

2. How many synthemes are there? Answer: There are 15 choices for the first duad in a syntheme and 6 for the second, since two of the numbers 1,2,3,4,5,6 are now used, so there are four left and there are six duads containing just these four. For example if the first duad is (12) then the choices for the second duad are (34), (35), (36), (45), (46), (56). This makes $15 \times 6 = 90$ but we have to allow for rearranging the order of the three duads in a syntheme. Any particular three duads can be arranged in six ways (three choices for the first duad, two for the second duad, one for the third duad). So we must divide 90 by 6 to give **15** synthemes.

Oh Boy! The same number of duads as synthemes!

3. Let's think about making synthemes which contain *different duads.* Let's start with
(12)(34)(56)
and make another syntheme not containing any of these duads. The new syntheme must pair 1 with something. Suppose it contains (13). How much further choice do we have? A little thought will convince you that there are exactly *two* possibilities, namely
(13)(26)(45) and (13)(25)(46)
These can be illustrated on a diagram, as follows.

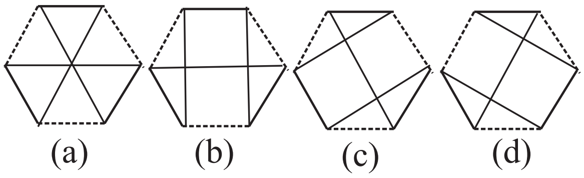


The solid lines are the first syntheme and the dotted lines are the second one. In each case we can redraw the figure so that it represents a single circuit of a regular hexagon!

Whatever two 'disjoint' synthemes (no duad in common) we start from we can always draw them as a circuit on a hexagon like this. All the remaining $15 - 6 = 9$ duads are then represented by diagonal lines across the hexagon.

4. Can the 15 duads be arranged in five rows of three so that every row is a syntheme? If so these five rows are referred to as a **total**.

The above diagram is a good start, for starting with say the synthemes
(12)(34)(56)
(13)(26)(45)
we want to add three more rows to this table. Each row will be represented on the diagram by three diagonal lines which all start and end at different corners of the hexagon. The only ways to draw three such diagonal lines are shown in the figure below.



We need to choose three of these which don't have any diagonal in common. Clearly we can only choose (b), (c), (d).

Completing the five synthemes starting with (12)(34)(56), (13)(26)45) we obtain in this way
(12)(34)(56)
(13)(26)(45)
(14)(25)(36)
(15)(23)(46)
(16)(24)(35)
which is indeed a total.

5. But, *even more important*, this shows that if we choose our first two synthemes, then there is *exactly one way* to complete these to a total by adding three more synthemes: two synthemes determine a total.

6. How many totals are there? This becomes relatively easy using the above result that we need only worry about the first two synthemes, and the rest will take care of themselves. Remember that in any total the order in which the five synthemes are written down does not matter.

Suppose we produce a total, as above, by writing down two synthemes having no duad in common. The total is then determined. There are 15 choices for the first syntheme. How many are there for the second syntheme? Consider the number 1. It can go in a duad with any of the numbers 2,3,4,5,6 *apart from* the one with which it was paired in the first syntheme. Thus it can go with four other numbers. (In the above example, 1 can be paired with 3,4,5,6.) For each of these there are exactly two ways of completing the syntheme—see **3.** above. So there are eight choices for the second syntheme. That makes 15×8 so far. But imagine we have the five synthemes written down, as in the example above. We could use any one of the five as the top row and any of the remaining four as the second row. So we must divide the number 15×8 by 5×4 to allow for this. That gives **6** totals.

Oh Boy! The number of totals is the same as the number of numbers 1,2,3,4,5,6 we started with!

So, in counting, **numbers = totals, duads = synthemes**. That is amazing. If you start with any number besides 6 it will not happen.

7. Choose a particular syntheme, e.g. (12)(34)(56). How many totals does this belong to? There are eight choices for the next syntheme in a total (see **6.** above). But we don't have to put it

as the second row, we could put it as any of the other four rows, and in every case the total is completely detemined—see **5.** above. So there will be $8/4 = $ **2 different totals containing the chosen syntheme**. For (12)(34)(56) these totals are the one above and
(12)(34)(56)
(13)(25)(46)
(14)(26)(35)
(15)(24)(36)
(16)(23)(45)

8. Also, **any two totals have exactly one syntheme in common**, for if they had two in common then the totals would be identical—see **5.** So from two totals we get one syntheme, in the same way that from two numbers we get one duad.

Let us call the six totals A,B,C,D,E,F. We can make a table, as follows. The totals can be read across or down, and the row and column corresponding to to different totals meet in their common syntheme. The totals in **4.** and **7.** above are A and B respectively, and the row and column meet in the syntheme (12)(34)(56).

	A	B	C	D	E	F
A		(12)(34)(56)	(13)(26)(45)	(14)(25)(36)	(15)(23)(46)	(16)(24)(35)
B	(12)(34)(56)		(15)(24)(36)	(16)(23)(45)	(14)(26)(35)	(13)(25)(46)
C	(13)(26)(45)	(15)(24)(36)		(12)(35)(46)	(16)(25)(34)	(14)(23)(56)
D	(14)(25)(36)	(16)(23)(45)	(12)(35)(46)		(13)(24)(56)	(15)(26)(34)
E	(15)(23)(46)	(14)(26)(35)	(16)(25)(34)	(13)(24)(56)		(12)(36)(45)
F	(16)(24)(35)	(13)(25)(46)	(14)(23)(56)	(15)(26)(34)	(12)(36)(45)	

9. Let's do the same with the totals that we did with the duads. Consider three pairs of totals—that is a 'syntheme of totals'—such as (AB)(CD)(EF). A and B have one syntheme in common (see **8.**), C and D have another one in common—it can't be the same syntheme as A and B since one syntheme belongs to exactly two totals, by **7.** Similarly E and F have exactly one syntheme in common, again a different one. So we end up with three different synthemes. For the example (AB)(CD)(EF) these three synthemes are (12)(34)(56), (12)(35)(46) and (12)(36)(45) respectively. All of these will contain a particular duad, in this case (12). The same applies to any 'syntheme of totals'. For instance (AE)(BC)(DF) gives the common syntheme (15)(23)(46) for A and E; (15)(24)(36) for B and C; and (15)(26)(34) for D and F, and all these contain the duad (15).

This means that every 'syntheme of totals' gives rise to a single duad. So we can make a second table reversing the first one. This time, a row and column giving a particular duad, such as (12), meet in the syntheme of totals, such as (AB)(CD)(EF). In the first table, the row and column for A,B meet in a syntheme containing (12), and similarly the row and column for C,D and for E,F.

	1	2	3	4	5	6
1		(AB)(CD)(EF)	(AC)(BF)(DE)	(AD)(BE)(CF)	(AE)(BC)(DF)	(AF)(BD)(CE)
2	(AB)(CD)(EF)		(AE)(BD)(CF)	(AF)(BC)(DE)	(AD)(BF)(CE)	(AC)(BE)(DF)
3	(AC)(BF)(DE)	(AE)(BD)(CF)		(AB)(CE)(DF)	(AF)(BE)(CD)	(AD)(BC)(EF)
4	(AD)(BE)(CF)	(AF)(BC)(DE)	(AB)(CE)(DF)		(AC)(BD)(EF)	(AE)(BF)(CD)
5	(AE)(BC)(DF)	(AD)(BF)(CE)	(AF)(BE)(CD)	(AC)(BD)(EF)		(AB)(CF)(DE)
6	(AF)(BD)(CE)	(AC)(BE)(DF)	(AD)(BC)(EF)	(AE)(BF)(CD)	(AB)(CF)(DE)	