Graphs, Trees and Brussels Sprouts

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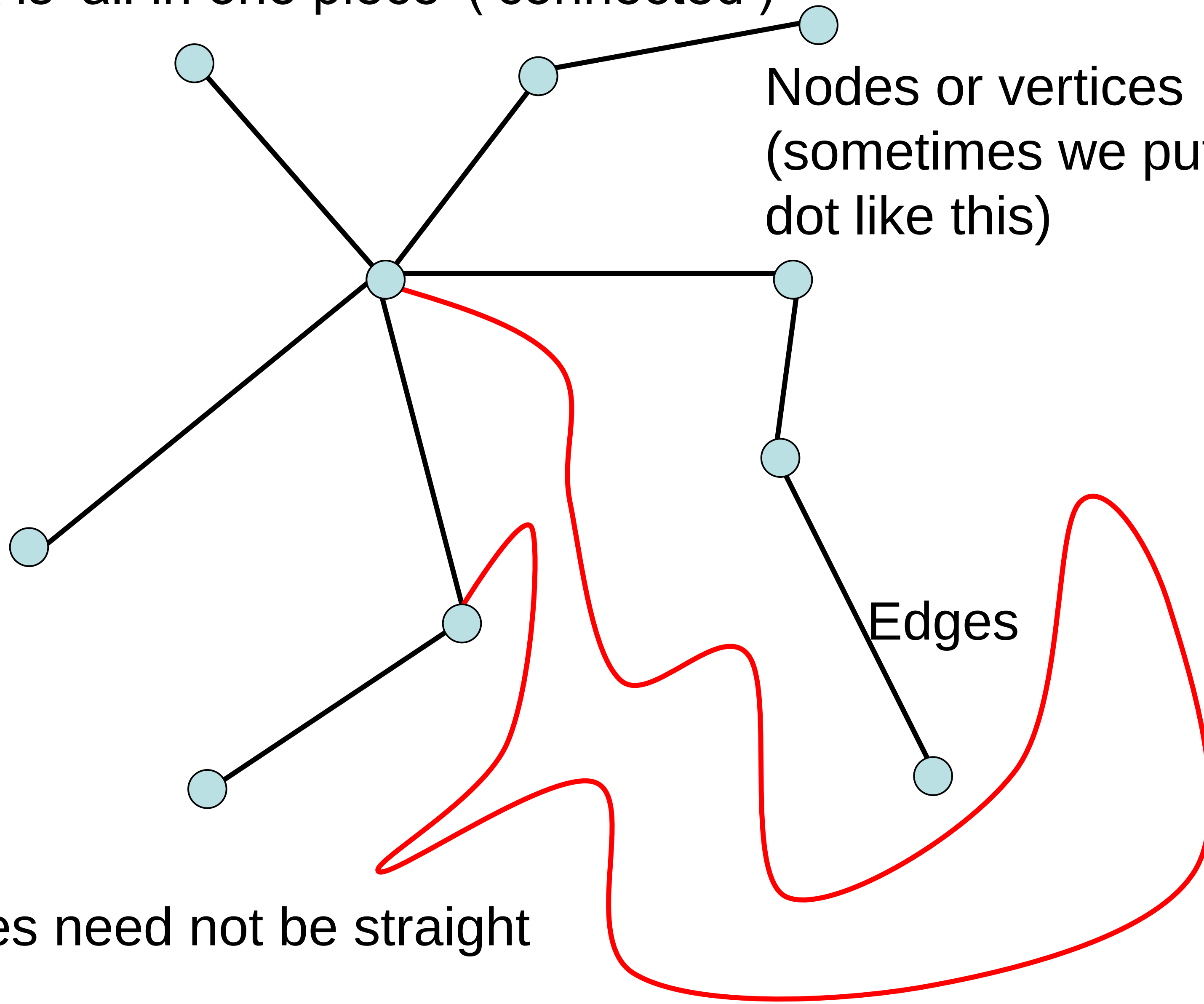
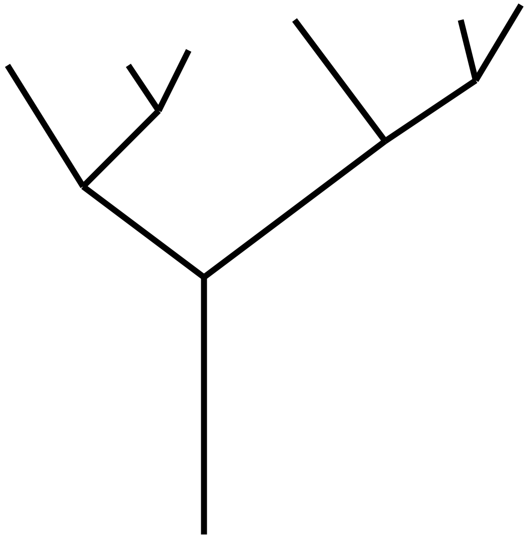
A tree has vertices and edges but no closed circuits.
However it is ‘all in one piece’ (‘connected’)



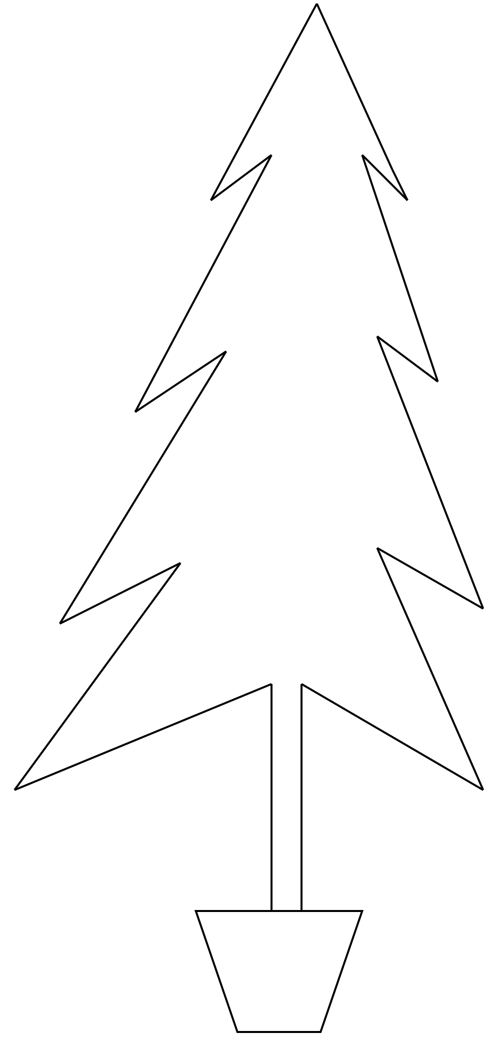
The edges need not be straight

but they should not cross except at a vertex

Which one is a tree?

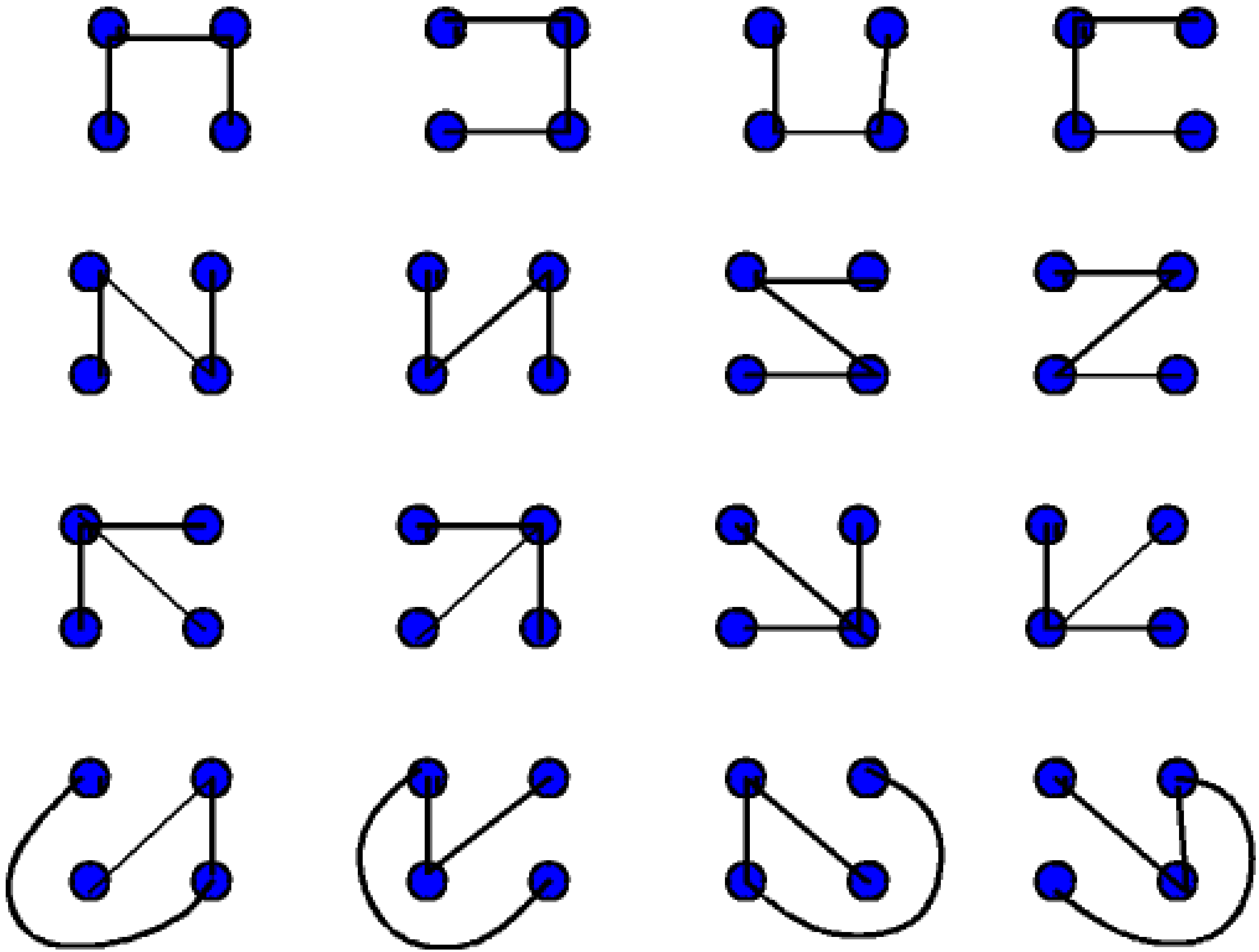


A tree



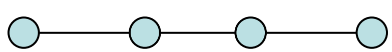
Not a tree

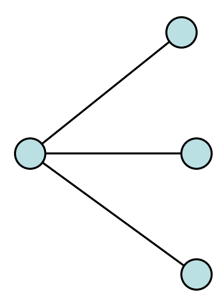
How many trees can you make with these four vertices?



The general formula (Cayley's formula) for the number of trees on n vertices is n^{n-2}, so for $n = 5$ it is already $5^3 = 125$.

It's harder to count the number of 'really different' trees, which for $n = 4$ is just 2:





v = number of vertices (nodes)

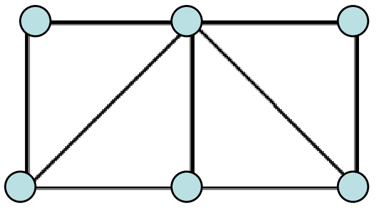
e = number of edges

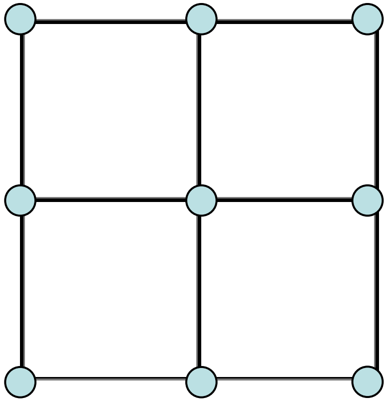
All these trees have $v = 4$, $e = 3$, so $v - e = 1$

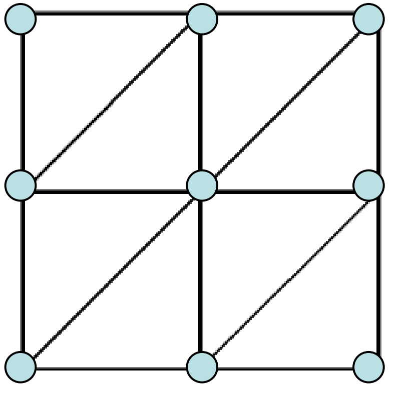
Try a couple more trees (any number of vertices)

Can you think of a reason why $v - e = 1$ is always true for any tree?

These graphs are not trees. How many edges must you remove to make them into trees? Remember that you don’t remove vertices, and the graph must remain in one piece.







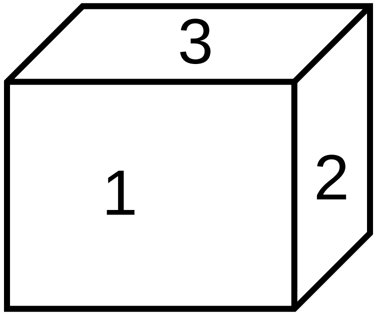
Even if two people don’t remove the same edges do they always remove the same **number of edges**?

The number of edges you must remove (without removing vertices) to make a tree is called the

cyclomatic number, μ (pronounced mew)

of the graph.

For each of the graphs which you just looked at count the number *r* of regions enclosed by the graph. For example with



there are three regions: $r = 3$.

What is the connection between *r* and μ ?

Euler's formula for a (connected) graph which is drawn in the plane:

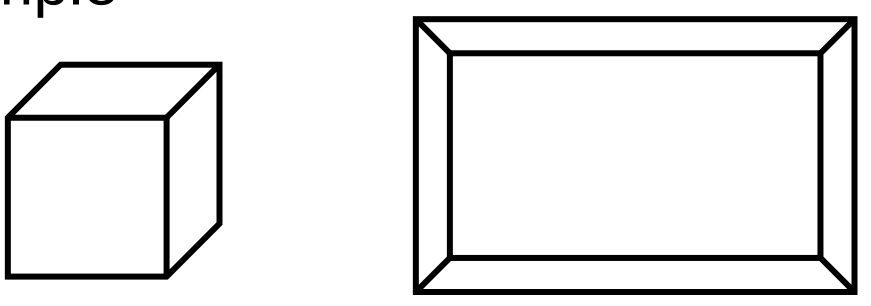
$v - e + r = 1$ (*r* is the number of regions **enclosed**)

Also *r* is the same as μ, the cyclomatic number, which is the number of edges you need to remove to turn the graph into a tree.

If the graph has *c* components (separate pieces) then

$v - e + r = c$

for example



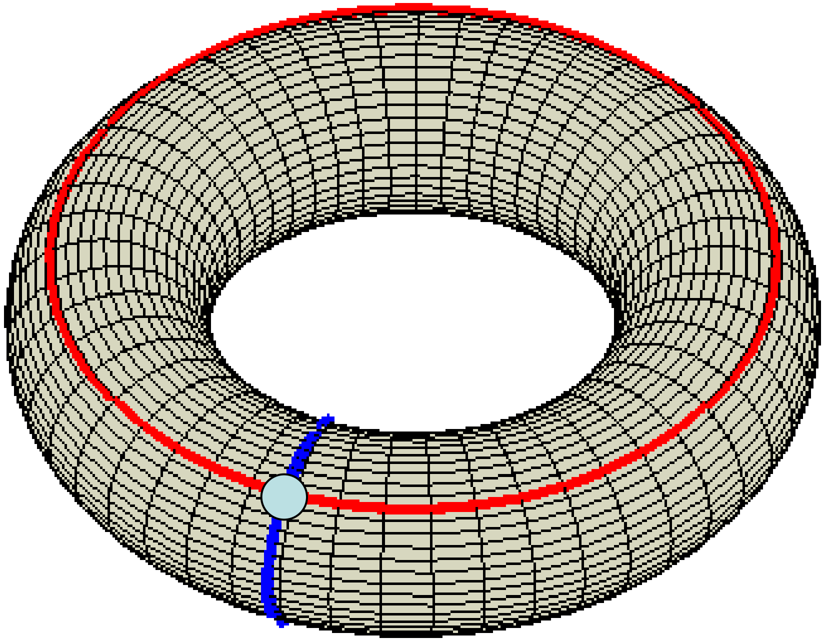
$v = 7 + 8 = 15$

$e = 9 + 12 = 21$

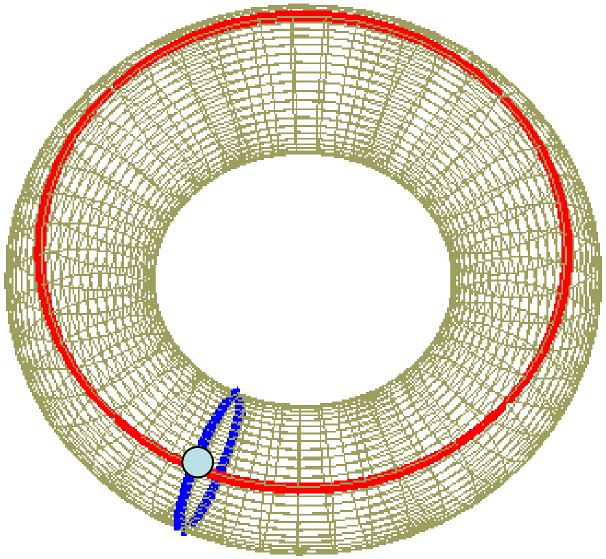
$r = 3 + 5 = 8$

$c = 2$

Let's look at one example NOT in the plane. This graph on the **torus** has *v* = 1, *e* = 2, and what is *r ???*

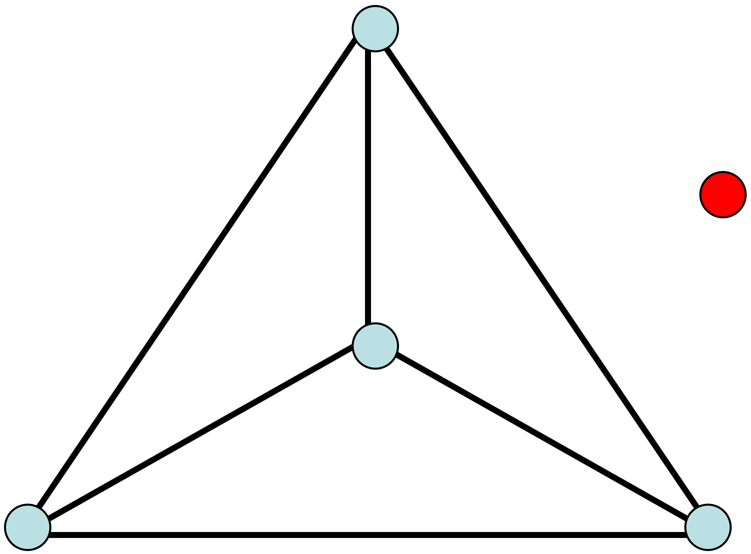


so here *v – e + r = ????*



a transparent view

This is called a **complete graph with four vertices:** every pair of vertices is joined by an edge



Do you think it is possible to add a fifth vertex and join it to all four existing vertices to make a complete graph with five vertices **drawn in the plane**?

(If we could then $v = 5$, $e = 10$, so $r = 1 + e - v = 6$,

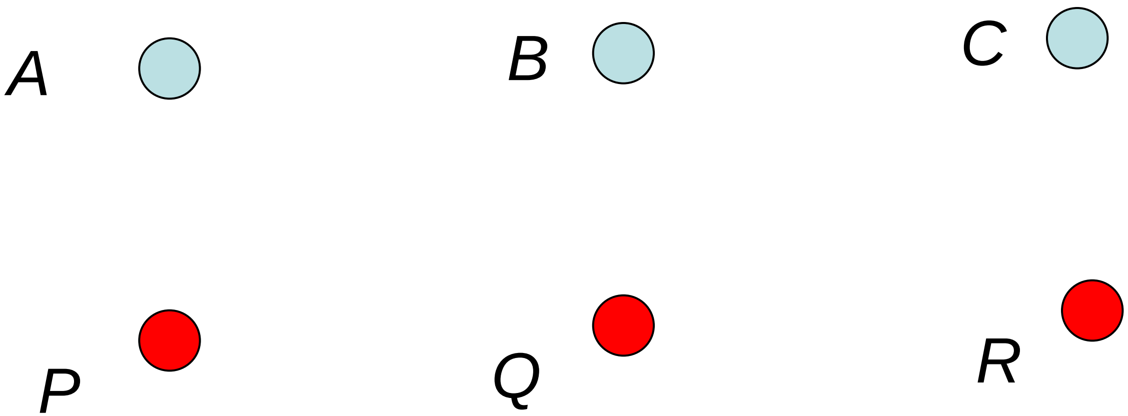
but every region must be surrounded by at least three edges, and every edge belongs to two regions, where here we also count the part of the plane ‘outside’ the graph. This means that

$2e \geq 3(r + 1)$ (the + 1 coming from the unbounded part of the plane ‘outside’ the graph)

so

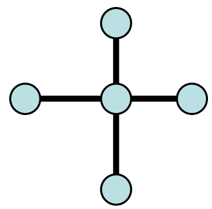
$20 \geq 21$, which is a contradiction.)

It's slightly harder to prove that you can't join each of *A, B, C* to each of *P, Q, R* (9 edges altogether) without making at least two of the edges cross:

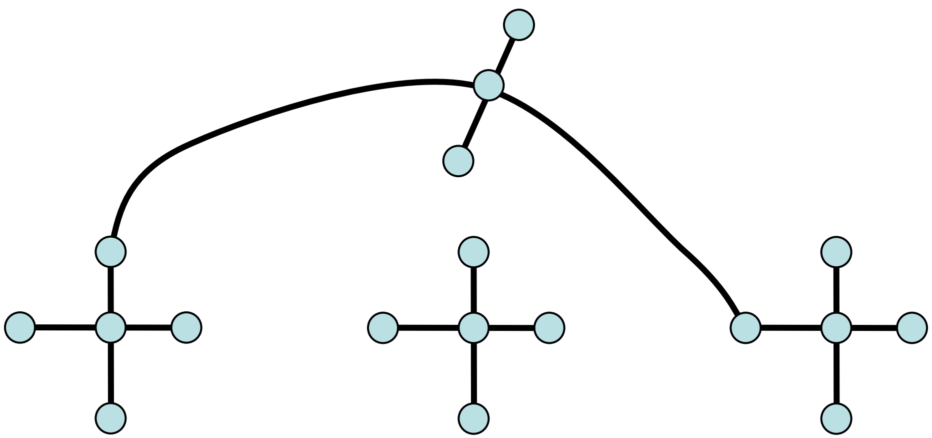


But finally what about the Brussels Sprouts?

This is a game invented, and named, by J.H.Conway.



Start with any number of crosses in the plane. The four spikes are called **sprouts.**



A move is to draw any line (not crossing other lines) joining two sprouts and put a crossing on it as shown. So 2 sprouts are lost and 2 gained.

We don’t really need to draw the vertices every time...

Two players move alternately.

The last person to have a valid move is the winner.

Every move increases e by 4 and v by 3 so after m moves

$v = 4n + 4m$, $e = 5n + 3m$,

but the number of available ‘sprouts’ stays constant at $4n$ throughout.

At the end of the game, the graph is *connected*, and there will be just one sprout pointing into every region (otherwise another move is possible), so the number of regions—including the ‘outside’ unbounded region—equals $4n$. So there are $r = 4n – 1$ enclosed regions.

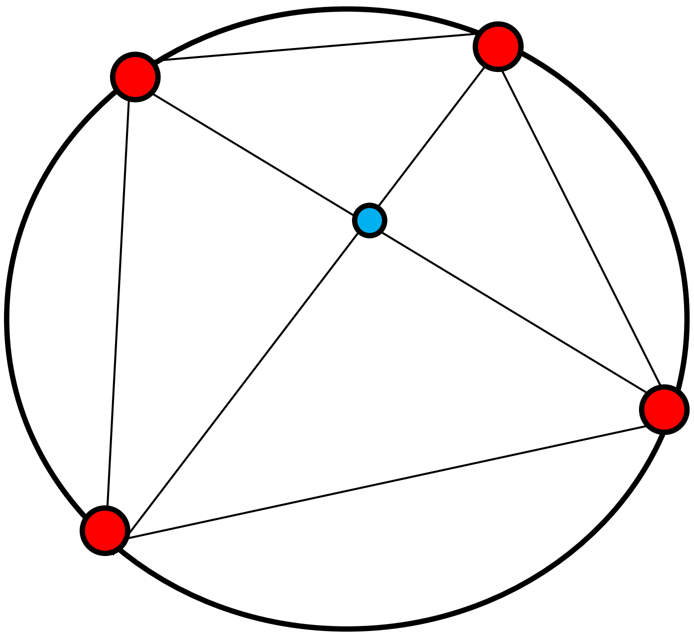
Using Euler’s formula, $v – e + r = 1$:

$5n + 3m – (4n + 4m) + 4n – 1 = 1$

which gives

$m = 5n – 2$: the number of moves is odd (player 1 wins) if and only if n is odd!

A problem for you to try.



n (red) points are taken round a circle (not generally evenly spaced) and every pair of points is joined by a segment. The figure shows *n* = 4. The *n* points are always spaced so that *three or more segments are never concurrent.*

Call every intersection of lines a vertex, and every segment between lines an edge, and also the *n* curved arcs of the circle. (In the figure there are 5 vertices, 12 edges.) For general *n* what are these numbers? Use Euler's formula to **deduce a formula for the number of regions inside the circle** (but first work it out from diagrams for *n* = 1, 2, 3, 4, 5).

Key words to look up on the internet:

Euler’s formula for graphs

counting trees

planar graphs

Kuratowski’s theorem on planar graphs

Conway’s game of sprouts