**Some more problems on Counting (Peter Giblin)**

(Starred problems are, perhaps, a shade more difficult)

If you succeed in solving some more at home, send to Peter Giblin, Department of Mathematical Sciences, The University of Liverpool, Liverpool L69 7ZL or pjgiblin@liv.ac.uk.

**1.**

$$f_n^2 + f_{n+1}^2 = f_{2n+2}$$

Look first at the case $n = 2$ : $f_2^2 + f_3^2 = f_6$.

Consider the middle point of the $6 \times 1$ board. There are two cases: (i) there is a domino across this place, (ii) there is no domino across this place: the tiling is 'breakable' at the middle.

Does this idea work for $n = 3$ : $f_3^2 + f_4^2 = f_8$?

In general, consider the middle point of the $(2n + 2) \times 1$ board, that is the right-hand end of the $(n + 1)^{\text{st}}$ cell of the board, which is a distance $n + 1$ from the left-hand end of the board. There are two cases: (i) there is a domino across this place, (ii) there is no domino across this place: the tiling is 'breakable' at the middle.

**2.**

$$\ell_n = f_{n-1} + 2f_{n-2}, \ n \geq 2$$

Split the tilings of an $n$-necklace into (i) those for which the tile covering Cell 1 is a square, (ii) those for which the tile covering Cell 1 is a domino.

Look at say $n = 4$ and $n = 5$ say, to begin with, to see what is going on. Can you show from this that $\ell_n = f_n + f_{n-2}$?

**3.**

$$\ell_n = \ell_{n-1} + \ell_{n-2}, \ n \geq 3 :$$

a similar relation to the one satisfied by the $f_n$ (and by the Fibonacci numbers).

Start by drawing the 4-necklaces and divide into two types, as follows. The *first tile* is the one covering the cell marked 1. The *last tile* is the tile immediately anticlockwise from the first tile. Now divide the 4-bracelets into (i) those for which the last tile is a square (how many of these?) and (ii) those for which the last tile is a domino (how may of these?). You might find that this fits well with the above formula!

Maybe the same idea works for $n = 5$? in general?

**4.** (This is a bit of algebra.)

From Problem 2 it is also possible to deduce that

$$\ell_n = \ell_{n-1} + \ell_{n-2}, \ n \geq 4.$$

Apply the formula of Problem 2 to $\ell_n, \ell_{n-1}$ and $\ell_{n-2}$ and use the property of the numbers $f_n$, namely $f_n = f_{n-1} + f_{n-2}, \ n \geq 2$. You can also deduce the formula above for $n = 3$ separately.)

**5*.** Explain why $f_{n+1}$ also counts the number of binary sequences of $n$ 0s and 1s ('binary $n$-tuples') in which there are no two consecutive 0s.

Look at the case $n = 3$, so the claim is that the count of tilings of a $4 \times 1$ board is the same as the count of binary 3-tuples where there are no two consecutive 0s. Try associating a square with 1 and a domino with 01. (This doesn't quite work but it is close. You can also use 'breakability' as in Problem 1.)

**6*.** Similarly $f_{n+1}$ counts the number of subsets of $\{1, 2, \ldots, n\}$ (including the empty set $\{ \ \}$ !) which do not contain two consecutive whole numbers.

For example, for $n = 3$ these subsets are $\{ \}, \{1, 3\}, \{1\}, \{2\}, \{3\}$, five in all.