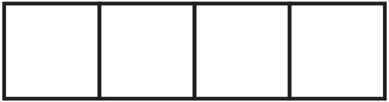
University of Liverpool Maths Club
November 2007

Peter Giblin

Counting

Find the five ways in which this 4×1 board can be tiled with squares and dominoes.



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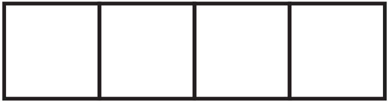

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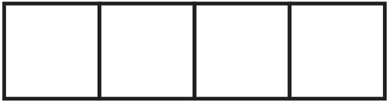

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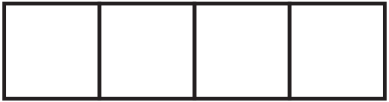

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So why is $f_4 = f_3 + f_2$?

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$$f_n = f_{n-1} + f_{n-2} \quad (n \geq 3) \ ?$$

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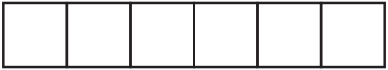
(The connexion with Fibonacci numbers F_n is that $f_{n-1} = F_n, n \geq 1$.)

Now consider this 6×1 board



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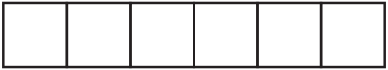

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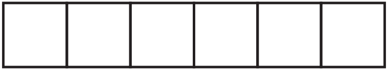

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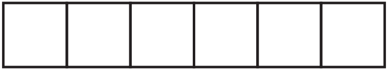

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How many ways?

Now let's count them a different way. Can someone **explain why** the number of ways is also

$$f_4 + f_3 + f_2 + f_1 + f_0 \quad \text{(remembering } f_0 = 1\text{)?}$$

(Think of where the *last domino* is placed!)

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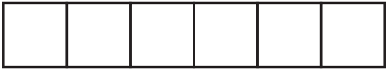

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So

$$f_4 + f_3 + f_2 + f_1 + f_0 = f_6 - 1.$$

The same argument proves this interesting

Theorem For any $n \geq 0$

$$f_n + f_{n-1} + f_{n-2} + \ldots + f_0 = f_{n+2} - 1.$$

This is an example of a theorem which can be proved by counting the same things in two different ways.

Now by considering tilings of a $2n \times 1$ board *using at least one square* see if you can **understand why**

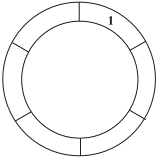
$$f_5 + f_3 + f_1 = f_6 - 1$$

(look at the position of the last square!)
and more generally why

$$f_{2n-1} + f_{2n-3} + \ldots + f_3 + f_1 = f_{2n} - 1,$$

with just the odd numbers appearing on the LHS.

A similar idea is **tiling of circular bracelets** such as this 6-bracelet:



with 'curved squares' and 'curved dominos'. We always call the first cell from the top towards the right 'Cell 1'. It might be covered by a curved square or, in two ways, by a curved domino.

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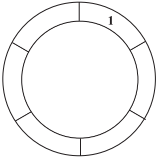

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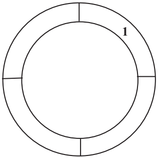
Write ℓ_n for the number of ways for an n-bracelet.

Find the seven ways in which a 4-bracelet can be tiled ($\ell_4 = 7$).



(These numbers are called Lucas numbers.)

1.

$$f_n^2 + f_{n+1}^2 = f_{2n+2}$$

Look first at the case $n = 2$: $f_2^2 + f_3^2 = f_6$.

Consider the middle point of the 6×1 board. There are two cases: (i) there is a domino across this place, (ii) there is no domino across this place: the tiling is 'breakable' at the middle.

Does this idea work for $n = 3$: $f_3^2 + f_4^2 = f_8$?

In general, consider the middle point of the $(2n + 2) \times 1$ board, that is the right-hand end of the $(n + 1)^{\mathrm{st}}$ cell of the board, which is a distance $n + 1$ from the left-hand end of the board. There are two cases: (i) there is a domino across this place, (ii) there is no domino across this place: the tiling is 'breakable' at the middle.

2.

$$\ell_n = f_{n-1} + 2f_{n-2}, \;\; n \geq 2$$

Split the tilings of an n-necklace into (i) those for which the tile covering Cell 1 is a square, (ii) those for which the tile covering Cell 1 is a domino.

Look at say $n = 4$ and $n = 5$ say, to begin with, to see what is going on. Can you show from this that $\ell_n = f_n + f_{n-2}$?

3.

$$\ell_n = \ell_{n-1} + \ell_{n-2}, \ \ n \geq 3:$$

a similar relation to the one satisfied by the f_n (and by the Fibonacci numbers).

Start by drawing the 4-necklaces and divide into two types, as follows. The *first tile* is the one covering the cell marked 1. The *last tile* is the tile immediately anticlockwise from the first tile. Now divide the 4-bracelets into (i) those for which the last tile is a square (how many of these?) and (ii) those for which the last tile is a domino (how may of these?). You might find that this fits well with the above formula!

Maybe the same idea works for $n = 5$? in general?

4.

(This is a bit of algebra.)

From Problem 2 it is also possible to deduce that

$$\ell_n = \ell_{n-1} + \ell_{n-2}, \;\; n \geq 4.$$

Apply the formula of Problem 2 to ℓ_n, ℓ_{n-1} and ℓ_{n-2} and use the property of the numbers f_n, namely $f_n = f_{n-1} + f_{n-2}, \;\; n \geq 2.$ You can also deduce the formula above for $n = 3$ separately.)

5*.

Explain why f_{n+1} also counts the number of binary sequences of n 0s and 1s ('binary n-tuples') in which there are no two consecutive 0s.

Look at the case $n = 3$, so the claim is that the count of tilings of a 4×1 board is the same as the count of binary 3-tuples where there are no two consecutive 0s. Try associating a square with 1 and a domino with 01. (This doesn't quite work but it is close. You can also use 'breakability' as in Problem 1.)

6*.

Similarly f_{n+1} counts the number of subsets of $\{1, 2, \ldots, n\}$ (including the empty set $\{\ \}$!) which do not contain two consecutive whole numbers.

For example, for $n = 3$ these subsets are $\{\ \}, \{1, 3\}, \{1\}, \{2\}, \{3\}$, five in all.