MATHEMATICS CLUB HANDOUT SHEET
MATRICES WITH MAPLE

In this session you will have the opportunity to learn something about **matrices** which arise frequently in mathematics when considering systems of **linear equations**. Linear equations are, as we shall see, generally easy to solve using matrix algebra. Consequently, one often uses the theory of linear systems to find approximate solutions to much more difficult **non-linear equations,** which occur in many branches of the physical sciences. The session will also introduce you to the computer algebra package Maple. Computer algebra packages, which can do many tedious and long winded algebraic manipulations and numerical calculations virtually instantaneously, are an increasingly important tool in all branches of mathematical research.

Basic Preliminaries
  A **matrix** is a rectangular array of real (or even complex) numbers. The numbers in the array are called the **entries** in the matrix. Some examples are listed below

$$\begin{bmatrix} 1 & 2 \\ 3 & 0 \\ -1 & 4 \end{bmatrix}, \begin{bmatrix} 2 & 1 & 0 & -3 \end{bmatrix}, \begin{bmatrix} -\sqrt{2} & \pi & 6.123 \\ 3 & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 9 \end{bmatrix}, \begin{bmatrix} 3 \end{bmatrix}$$

As these examples indicate, matrices vary in size. The **size** of a matrix is described by specifying the number of **rows** (horizontal lines) and **columns** (vertical lines) that occur in the matrix. The first matrix has 3 rows and two columns so its size is 3 by 2 (written 3×2). The first number indicates the number of rows and the second the number of columns. Can you write down the sizes of the other four matrices listed above?
Size of matrix 2=   Size of matrix 3=   Size of matrix 4=   Size of matrix 5=

Conventionally matrices are usually denoted by capital letters, whilst small letters are often used to denote the entries. Suppose we had a 2×3 matrix denoted by $A = \begin{bmatrix} 2 & 1 & 7 \\ 3 & -0.567 & 1/3 \end{bmatrix}$. If A is a general matrix, we use the letter a_{ij} to denote the entry that occurs in row i and column j. So our matrix would be

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix},$$

where $a_{11} = 2$, $a_{12} = 1, a_{13} = 7, a_{21} = 3$ $a_{22} = -0.567$ and $a_{23} = 1/3$ in this case.

Two matrices A and B are said to be equal, denoted by $A = B$, if they are the **same size** and possess **identical entries.**

Let us see how we can input the matrix A into Maple. On your computer screen you should see an icon denoted "Maple 10 Classic..." Point the cursor at the icon and click

to start Maple. You should see the symbol ">" at the beginning of a blank worksheet. This is called the Maple prompt.

At the prompt type **A:=Matrix([[2,1,7],[3,-10,-2]]);** Then press **[Enter]**

(The semi colon is very important! It ends all Maple commands.)

You should see

$$A := \begin{bmatrix} 2 & 1 & 7 \\ 3 & -10 & -2 \end{bmatrix} \quad \text{appear on the screen.}$$

In this way Maple creates and stores the object A as 2×3 matrix. If you wanted to

input the 3×3 matrix $B = \begin{bmatrix} 3 & 1 & -1/10 \\ 0 & 1 & 6 \\ -0.632 & 2 & -92/5 \end{bmatrix}$, one would type in

B:=Matrix([[3,1,-1/10],[0,1,6],[-0.632,2,-92/5]]); [Enter]

Addition and Subtraction

   Matrices can added and subtracted, **provided they are of the same size.** All one does is add, or subtract, the corresponding entries.

So if A is the matrix above and C is the matrix $\begin{bmatrix} -4 & 2 & -6 \\ 1 & 0 & 8 \end{bmatrix}$ then

$$A + C = \begin{bmatrix} 2+(-4) & 2+1 & 7+(-6) \\ 3+1 & -10+0 & -2+8 \end{bmatrix} = \begin{bmatrix} -2 & 3 & 1 \\ 4 & -10 & 6 \end{bmatrix},$$

whilst

$$A - C = \begin{bmatrix} 2-(-4) & 2-1 & 7-(-6) \\ 3-1 & -10-0 & -2-8 \end{bmatrix} = \begin{bmatrix} 6 & 1 & 13 \\ 2 & -10 & -10 \end{bmatrix}.$$

Let us see how Maple handles this. Input the matrix C by typing **C:=Matrix([[-4,2,-6],[1,0,8]]);** [Enter]

Now you can add and subtract matrices using the **eval** command. Type in **eval(A+C); [Enter]** and you should see the result of A+C. Likewise typing in **eval(A-C); [Enter]** gives you A-C.

Consider the following matrix $E = \begin{pmatrix} 1 & 1 & 3 \\ 4 & 6 & -2 \\ 0 & 1 & 8/5 \end{pmatrix}$ and the matrices A and B above.

Can you work out.

i) B+E,   ii) A+E   iii) E-B?

Do the calculations long hand and then get Maple to check your results. (N.B. the letter D is protected in Maple, so do not try to assign a matrix to it.)

Matrix Multiplication

If we multiply a matrix A by a scalar c then the **product** cA is the matrix obtained by multiplying each entry of A by c.

So for instance if $A = \begin{bmatrix} 4 & 2 \\ 1 & 3 \\ -1 & 0 \end{bmatrix}$, then $4A = \begin{bmatrix} 4 \times 4 & 4 \times 2 \\ 4 \times 1 & 4 \times 3 \\ 4 \times (-1) & 4 \times 0 \end{bmatrix} = \begin{bmatrix} 16 & 8 \\ 4 & 12 \\ -4 & 0 \end{bmatrix}$.

Matrix Multiplication is a little more complicated.

If A is an $m \times r$ matrix and B is an $r \times m$ matrix, then the **product** AB is $m \times n$ matrix whose entries are determined as follows. To find the entry in row i and column j of AB, single out row i from the matrix A and column j from the matrix B. Multiply the corresponding entries from the row and column together and add up the resulting products.

Consider the matrices

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 4 & 1 & 4 & 3 \\ 0 & -1 & 3 & 1 \\ 2 & 7 & 5 & 2 \end{bmatrix}.$$

Since A is 2×3 and B is 3×4, the product AB is a 2×4 matrix. To determine, for example, the entry in row 2 and column 3 of AB, we single out row 2 from A and column 3 from B. Then as illustrated below, we multiply corresponding entries together and add up the products.

$$\text{row } 2 \rightarrow \begin{bmatrix} 1 & 2 & 4 \\ (2 & 6 & 0) \end{bmatrix} \begin{bmatrix} 4 & 1 & \overset{\cap}{4} & 3 \\ 0 & -1 & 3 & 1 \\ 2 & 7 & \underset{\cup}{5} & 2 \end{bmatrix} = \begin{bmatrix} * & * & * & * \\ * & * & 26 & * \end{bmatrix}$$

$$\text{col } 3$$

$$(2 \times 4) + (6 \times 3) + (0 \times 5) = 26$$

The entry in row 1 and column 4 of AB is computed as follows:

$$\text{row } 1 \rightarrow \begin{bmatrix} (1 & 2 & 4) \\ 2 & 6 & 0 \end{bmatrix} \begin{bmatrix} 4 & 1 & 4 & \overset{\cap}{3} \\ 0 & -1 & 3 & 1 \\ 2 & 7 & 5 & \underset{\cup}{2} \end{bmatrix} = \begin{bmatrix} * & * & * & 13 \\ * & * & 26 & * \end{bmatrix}$$

$$\text{col } 4$$

$$(1 \times 3) + (2 \times 1) + (4 \times 2) = 13$$

We can compute the remaining products as follows:

$$\left(1\times4\right)+\left(2\times0\right)+\left(4\times2\right)=12$$
$$\left(1\times1\right)+\left(2\times-1\right)+\left(4\times7\right)=27$$
$$\left(1\times4\right)+\left(2\times3\right)+\left(4\times5\right)=30$$
$$\left(2\times4\right)+\left(6\times0\right)+\left(0\times2\right)=8$$
$$\left(2\times1\right)+\left(6\times-1\right)+\left(0\times7\right)=-4$$
$$\left(2\times3\right)+\left(6\times1\right)+\left(0\times2\right)=12$$

$$AB=\begin{bmatrix}12 & 27 & 30 & 13\\ 8 & -4 & 26 & 12\end{bmatrix}.$$

Let's see how maple does this calculation. Define the matrices A and B in the standard way
A:=Matrix([[1,2,4],[2, 6, 0]]); [enter]
B:=Matrix([[4,1,4, 3],[0, -1, 3, 1], [2, 7, 5, 2]]); [enter]
AB:=A.B; [enter]

You should see the answer come as above.
A very important point about matrix multiplication is that the number of columns of the first matrix must equal the number of rows of the second, otherwise it is invalid. This means that unlike ordinary multiplication of numbers, where $a\times b=b\times a$, $AB\neq BA$. Matrix multiplication is **non-communative**.
You can see this by typing in **BA:=B.A; [enter]** in Maple. The result is an error message because the number of columns of B does not equal the number of rows of A.

Here are five matrices. Can you work out the products, if they exist, given below? Use Maple to check your answers.

$$A=\begin{bmatrix}3 & 0\\ -1 & 2\\ 1 & 1\end{bmatrix}, \quad B=\begin{bmatrix}4 & -1\\ 0 & 2\end{bmatrix}, \quad C=\begin{bmatrix}1 & 4 & 2\\ 3 & 1 & 5\end{bmatrix}, \quad D=\begin{bmatrix}1 & 5 & 2\\ -1 & 0 & 1\\ 3 & 2 & 4\end{bmatrix}, \quad E=\begin{bmatrix}6 & 1 & 3\\ -1 & 1 & 2\\ 4 & 1 & 3\end{bmatrix}$$

What are: a) AB, b) DE, c) ED, d) $(AB)C$, e) $A(BC)$, f) $D+E^2$ where $E^2=EE$?

Linear Equations and Matrices

Linear equations are simply equations that are linear in the unknowns x and y (that is they are raised to a power 1). Here is a very simple example.

$$6x + 4y = 24$$
$$4x - 5y = -7$$

The two unknowns, x and y, satisfy two linear equations. Can you solve these equations and find x and y? Solution $x =$ $y =$

How can matrix algebra be useful in solving equations such as this (and much larger sets of equations)? Well with our knowledge of matrix algebra, we can write the above set of equations in the form

$$A\boldsymbol{x} = \boldsymbol{b} \quad \equiv \quad \begin{bmatrix} 6 & 4 \\ 4 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 24 \\ -7 \end{bmatrix},$$

where A is a 2×2 square matrix, $\boldsymbol{x} = \begin{bmatrix} x \\ y \end{bmatrix}$ and $b = \begin{bmatrix} 24 \\ -7 \end{bmatrix}$ are 2×1 matrices, or column

vectors.

How does this help? Well let us suppose A, $\boldsymbol{x}$ and $\boldsymbol{b}$ were just ordinary real numbers and we wanted to find $\boldsymbol{x}$ such that $A\boldsymbol{x} = \boldsymbol{b}$. Provided $A \neq 0$, we would multiply both

sides of the equation by the reciprocal of A, that is $\dfrac{1}{A} = A^{-1}$ to give

$A^{-1}A\boldsymbol{x} = A^{-1}\boldsymbol{b} \Rightarrow 1 \times \boldsymbol{x} = A^{-1}\boldsymbol{b}$. All very obvious, all though in our solution we would not bother with the one times bit. The solution to the matrix equation is analogous. We multiply both sides by a **matrix** to give the equivalent of "one times" on the left hand side. But the "one times" is not a number but a square matrix of the simplest

form, viz. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. That is a matrix with ones on the main diagonal top left to lower

right and zeros elsewhere. So what we are looking for is a matrix, we will denote by

A^{-1}, so that $A^{-1}A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. *If such a matrix can be found* then in our example above

we multiply both sides by A^{-1}, giving

$$A^{-1}A\boldsymbol{x} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \boldsymbol{x} = \begin{bmatrix} x \\ y \end{bmatrix} = A^{-1}\boldsymbol{b}.$$

The right hand side is just a 2×1 column vector giving the answers to x and y.

The matrix A^{-1} is called the **inverse** matrix of A. The matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is called the

2×2 **identity matrix** and is denoted by I. NB. $IA = AI = A$, $\quad A^{-1}A = AA^{-1} = I$.

Now for 2×2 matrices, it is relatively easy to calculate the inverse matrix. Suppose we have a general 2×2 matrix

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then the corresponding inverse is given by $A^{-1} = \dfrac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$,

provided that $ad - bc \neq 0$. Check out this result by multiplying A by A^{-1}. The number $ad - bc$, is called the **determinant** of the matrix, and is denoted by det A. Inverse matrices only exist if det $A \neq 0$.

So in our example above $A = \begin{bmatrix} 6 & 4 \\ 4 & -5 \end{bmatrix}$, det $A = 6 \times (-5) - 4 \times 4 = -46 \neq 0$. Hence the

inverse exists, and we can write it down in the from $A^{-1} = \dfrac{1}{-54} \begin{bmatrix} -5 & -4 \\ -4 & 6 \end{bmatrix}$.

Multiplying through by A^{-1} on both sides of the equation gives

$$\frac{1}{-46}\begin{bmatrix} -5 & -4 \\ -4 & 6 \end{bmatrix}\begin{bmatrix} 6 & 4 \\ 4 & -5 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{-46}\begin{bmatrix} -46 & 0 \\ 0 & -46 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} =$$
$$\frac{1}{-46}\begin{bmatrix} -5 & -4 \\ -4 & 6 \end{bmatrix}\begin{bmatrix} 24 \\ -7 \end{bmatrix} = \frac{1}{-46}\begin{bmatrix} -120 + 28 \\ -96 - 42 \end{bmatrix} = \frac{1}{-46}\begin{bmatrix} -92 \\ -138 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

Hence $x=2$ and $y=3$ as you no doubt found earlier.
We can use maple to do these calculations.
First input **with(LinearAlgebra);**
This loads a number of clever subroutines into Maple for performing matrix operations. You only need do this once. Now type

A:=Matrix([[6,4], [4, -5]]);

The determinant can be found by typing **Determinant(A);;**

whilst the inverse can be found by typing **Ain:= MatrixInverse(A);**

Finally type in **b:=Matrix([[24],[-7]]);** followed by **xy:=Ain.b;** which gives the answers we are looking for.

Can you solve the following 2×2 systems of equations? First find the solutions long hand, then input the matrices in maple, calculate the determinants and inverse matrices and finally x and y.

a) $\begin{aligned} x + y &= 7 \\ 2x + 5y &= -3 \end{aligned}$
b) $\begin{aligned} 3x - 6y &= 8 \\ 2x + 5y &= 1 \end{aligned}$

Calculating determinants and inverse matrices for 3×3 matrices and solving the corresponding sets of equations are considerably more difficult. However, use of Maple makes things much more tractable. Can you solve the following sets of linear equations?

$$x + 2y + 2z = -1 \qquad\qquad 2x + y + z = 7$$

a) $x + 3y + z = 4$ b) $3x + 2y + z = -3$ c)

$$x + 3y + 2z = 3 \qquad\qquad 0x + y + z = 5$$

$$3w + x + 7y + 9z = 4$$
$$w + x + 4y + 4z = 7$$
$$-w + 0x - 2y - 3z = 0$$
$$-2w - x - 4y - 6z = 6$$

To do this
- i)   write systems in matrix form $A\,x = b$
- ii)   type in the matrix A into Maple
- iii)   calculate its determinant and check it is non zero.
- iv)   Get Maple to calculate A^{-1} and store in **Ain.**
- v)   Multiply the column vector on the right hand side by **Ain,** to get the solutions to x, y, z (and w)

For the very speedy amongst you.

A very interesting situation occurs when we have a system of equations $A\,x = 0$, when $\det A = 0$. Obviously if $\det A \neq 0$, the inverse would exist and the only solution would be $x=0$. However if $\det A = 0$ it is sometimes possible to find non zero solutions to this type of system.

Here is a special situation. Let consider the matrix $\begin{bmatrix} 3 & 2 \\ -1 & 0 \end{bmatrix}$, and consider solutions of the following system of equations.

$$\begin{bmatrix} 3 & 2 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \lambda \begin{bmatrix} x \\ y \end{bmatrix},$$

where λ is an unknown real number. We can rewrite this system in the form

$$\begin{bmatrix} 3 & 2 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} \Rightarrow \left(\begin{bmatrix} 3 & 2 \\ -1 & 0 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right)\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 3-\lambda & 2 \\ -1 & -\lambda \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now the latter expression is in the form $A\,x = 0$ and if $A = \begin{bmatrix} 3-\lambda & 2 \\ -1 & -\lambda \end{bmatrix}$ has non zero determinant, A^{-1} exists, and **x=0.** But what happens if we were to choose λ so that $\det A = 0$? Remember for a 2×2 matrix the determinant is given by $ad - bc$. So in this case we would require

$$(3-\lambda)(-\lambda) - 2(-1) = \lambda^2 - 3\lambda + 2 = (\lambda - 1)(\lambda - 2) = 0.$$

So for the special values $\lambda = 1$ and $\lambda = 2$ $\det A = 0$. These special values are called the **eigenvalues** of the matrix. Substituting $\lambda = 1$ and $\lambda = 2$ into the original equations gives

$$\begin{bmatrix} 3 & 2 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 1\begin{bmatrix} x \\ y \end{bmatrix} \Rightarrow y = -x \text{ with } x \text{ any real number}$$

$$\begin{bmatrix} 3 & 2 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 2\begin{bmatrix} x \\ y \end{bmatrix} \Rightarrow 2y = -x, \text{ with } x \text{ any real number}$$

If we choose $x = 1$ in each case, we have two non-zero solutions to our system of equations, viz. $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ for $\lambda = 1$, and $\begin{bmatrix} 1 \\ -1/2 \end{bmatrix}$ for $\lambda = 2$. These non-zero solutions of the system of equations are called the **eigenvectors** associated with the particular eigenvalues. Notice because we can choose any value for x, any multiple of these vectors is also an eigenvector.

Now form the "matrix of eigenvectors" $P = \begin{pmatrix} 1 & 1 \\ -1 & -1/2 \end{pmatrix}$, and use Maple to calculate the matrix $E = P^{-1}\begin{bmatrix} 3 & 2 \\ -1 & 0 \end{bmatrix}P$ (or by long hand if you prefer). What do you notice about E? Can you use your expression for E to find $\begin{bmatrix} 3 & 2 \\ -1 & 0 \end{bmatrix}$ raised to the power 5 (that is multiplied by itself 5 times) very easily? Explain your result and check using Maple.

Exercise

Here is a 3×3 matrix $C = \begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$. Can you find its (three) eigenvalues and (three) corresponding eigenvectors. This is much trickier than the 2×2 case above, but with Maple it is easy.
Use Maple to find the eigenvalues by typing **Eigenvalues(C);**
To find the eigenvectors type **Eigenvectors(C, output=list);**

Form the "matrix of eigenvectors" P as above and calculate $E = P^{-1}CP$. What do you notice? Again use this expression to find a simple way of calculating C raised to the power of 5.

The eigenvalues and associated eigenvectors are extremely important characteristics, which allow matrices to be broken down and represented in very simple forms. If you pursue the subject to a higher level, you will find eigensystems of matrices turn up in many different applications of mathematics. Consequently the theory underlying this work forms an important branch of study in the subject.