Liverpool University Mathematics Club

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Introduction: Matrices

Matrices are a bit like coordinates (eg. $(6,4)$) or vectors (eg. $\begin{pmatrix} -5 \\ 1 \end{pmatrix}$) in that they are an array of numbers. For example: $\begin{pmatrix} 5 & 3 & 2 \\ 1 & -5 & 0 \end{pmatrix}$ is called a 2×3 matrix. But matrices can be of any size.

Matrices can be added and subtracted naturally: for example

$$\begin{pmatrix} 1 & 5 \\ 8 & 7 \end{pmatrix} + \begin{pmatrix} 3 & 2 \\ -11 & 0 \end{pmatrix} = \begin{pmatrix} 1+3 & 5+2 \\ 8+(-11) & 7+0 \end{pmatrix} = \begin{pmatrix} 4 & 7 \\ -3 & 7 \end{pmatrix}$$

The result is always a matrix the same size as the two (or three or any number) added together and each element (each number) is just the sum of the two in the same place as it.

Similarly for subtraction: for example: $\begin{pmatrix} 4 & 7 \end{pmatrix} - \begin{pmatrix} 2 & 3 \end{pmatrix} = \begin{pmatrix} 4-2 & 7-3 \end{pmatrix} = \begin{pmatrix} 2 & 4 \end{pmatrix}$.

You can multiply any matrix by a number: you just multiply every element by it, eg.

$$2\begin{pmatrix} 5 & 2 \\ 4 & 1 \end{pmatrix} = \begin{pmatrix} 2\times 5 & 2\times 2 \\ 2\times 4 & 2\times 1 \end{pmatrix} = \begin{pmatrix} 10 & 4 \\ 8 & 2 \end{pmatrix}$$

The **zero matrix** is a square matrix with 0's in every position, eg. $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. It is called the zero matrix because it acts just like a zero. For instance, looking addition and multiplication (by a number), if we label a zero matrix by $\mathbf{0}$ then for any other matrix (of the same size) $\mathbf{M}$ then $\mathbf{M}+\mathbf{0}=\mathbf{M}$ and for any number λ then $\lambda\mathbf{0}=\mathbf{0}$. We will see a little more of this matrix later

Some Questions

1. Find $\begin{pmatrix} 3 & 5 & -1 & 0 \\ 2 & 6 & 0 & 2 \end{pmatrix} + \begin{pmatrix} 7 & 5 & 10 & 2 \\ 1 & 6 & 2 & 3 \end{pmatrix}$
2. If $\mathbf{A} = \begin{pmatrix} 7 & 1 \\ 0 & -y \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 4 & 3 \\ 2 & x \end{pmatrix}$ then find x, y, z if

$$\mathbf{A}+5\mathbf{B}=\begin{pmatrix}27 & z\\ 10 & 18\end{pmatrix} \text{ and } 2\mathbf{A}-\mathbf{B}=\begin{pmatrix}10 & -1\\ -2 & -19\end{pmatrix}.$$

3. How many possible pairs of matrices $\mathbf{x},\mathbf{y}$ (with only whole number elements above zero) exist such that $\mathbf{x}+\mathbf{y}=\begin{pmatrix}\mathbf{7} & 2 & 10\end{pmatrix}$.

Matrix Multiplication

Two matrices can also sometimes be multiplied together. For example:

$$\underbrace{\begin{pmatrix}1 & 0 & 2\\ 2 & 3 & 5\end{pmatrix}}_{2 \text{ rows}}\underbrace{\begin{pmatrix}1 & 2 & -1\\ 4 & 3 & 0\\ 0 & 1 & 5\end{pmatrix}}_{3}=\underbrace{\begin{pmatrix} & & \\ & & \end{pmatrix}}_{3}\Big\}2$$

The product has the height of the first and the width of the second.

To find the element on the first (n th) row and first (m th) column, you look at the numbers in the first (n th) row of the first matrix and the first (m th) column of the second: you multiply the first of each of the row and column together, and then the second and then the third, and then add these together:

$$\begin{pmatrix}1 & 0 & 2\\ 2 & 3 & 5\end{pmatrix}\begin{pmatrix}1 & 2 & -1\\ 4 & 3 & 0\\ 0 & 1 & 5\end{pmatrix}=\begin{pmatrix}(1\times1)+(0\times4)+(2\times0) & & \\ & & \end{pmatrix}$$

You do likewise for every element…

$$\begin{pmatrix}1 & 0 & 2\\ 2 & 3 & 5\end{pmatrix}\begin{pmatrix}1 & 2 & -1\\ 4 & 3 & 0\\ 0 & 1 & 5\end{pmatrix}=\begin{pmatrix}1\times1+0\times4+2\times0 & 1\times2+0\times3+2\times1 & 1\times(-1)+0\times0+2\times5\\ 2\times1+3\times4+5\times0 & 2\times2+3\times3+5\times1 & 2\times(-1)+3\times0+5\times5\end{pmatrix}=\begin{pmatrix}1 & 4 & 9\\ 14 & 18 & 23\end{pmatrix}$$

This may seem a strange way to multiply, but there's a very good reason for it, which we will look at later.

Notice that any product of a matrix and the zero matrix is the zero matrix, just like the zero of numbers.

Also, there is another special square matrix called the **identity** or **unit** matrix that works like the number 1 for multiplication: any matrix multiplied by it is unchanged. This matrix has 1's along the main diagonal and 0's everywhere else, eg. $\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$, $\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$. For example, $\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}5 & 1\\ 3 & -9\end{pmatrix}=\begin{pmatrix}1\times5+0\times3 & 1\times1+0\times(-9)\\ 0\times5+1\times3 & 0\times1+1\times(-9)\end{pmatrix}=\begin{pmatrix}5 & 1\\ 3 & -9\end{pmatrix}$.

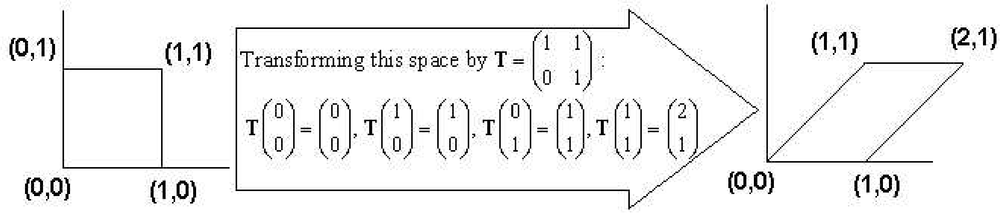
Some More Questions

1. Find $\begin{pmatrix}5 & 7\\ 1 & 6\end{pmatrix}\begin{pmatrix}1 & 0 & 6\\ 3 & 1 & 2\end{pmatrix}$.
2. Where $\mathbf{C}=\begin{pmatrix}91 & 0\\ 0 & -8\end{pmatrix}, \mathbf{Z}=\begin{pmatrix}x & 0\\ 0 & y\end{pmatrix}$ find x, y where $\mathbf{Z}^2-6\mathbf{Z}-\mathbf{C}=\mathbf{0}$ ($\mathbf{Z}^2=\mathbf{ZZ}$).
3. *When can two matrices be multiplied?
4. *How could you write the system of equations: $3x+5y+z=17$, $6x-4y-z=1$, $-6x+18y+4z=32$ as $\mathbf{AB}=\mathbf{C}$ where $\mathbf{A}, \mathbf{C}$ are constants?

Linear Transformations

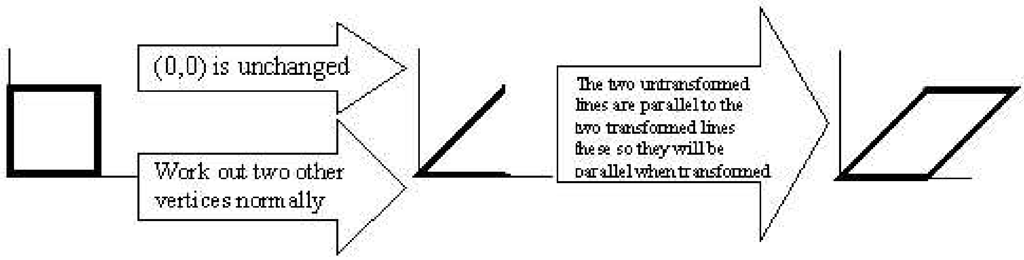
In a space, the coordinates (x, y) can be treated as 2×1 matrices: $\begin{pmatrix} x \\ y \end{pmatrix}$.

When these are multiplied by a constant matrix, the space is *linearly transformed*. For example the following transformation is a *shear*:



Linear transformations do not simply move vertices and join up the lines between them. Linear transformations transform every single infinitesimal point in space but you only need to find the positions of the vertices (because if you transform all the points on a line you get another line).

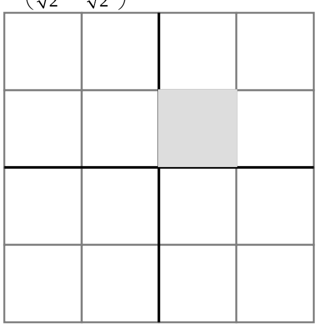
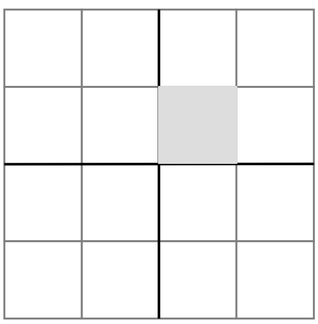
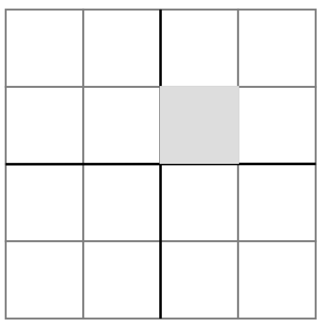
There are other shortcuts: linear transformations never move the point $(0,0)$ so you don't need to work out where this point is every time; if there are two parallel lines originally then after the transformation they will remain parallel. So using these two shortcuts the above shear is found far quicker:



Exercises

1. Try working out the following transformations of the unit square (draw them on the same graph) and try stating what type of transformation they are:

$\begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$ $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

2. a). Transform the unit square with $\mathbf{A}=\begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix}$.

   b). Transform the resultant shape with $\mathbf{B}=\begin{pmatrix} 0.75 & 0.5 \\ 1 & 1 \end{pmatrix}$

   *c). Transform the unit square with $\mathbf{BA}$. What do you notice?

3. *Prove that a Linear Transformation transforms lines into lines. (Also, prove that a pair of parallel lines remains parallel after a transformation.)

4. *The definition of a Linear Transformation is any function $\mathbf{T}(\mathbf{x})$ such that $\mathbf{T}(\lambda\mathbf{x})=\lambda\mathbf{T}(\mathbf{x})$ and $\mathbf{T}(\mathbf{x}+\mathbf{y})=\mathbf{T}(\mathbf{x})+\mathbf{T}(\mathbf{y})$. Show that $\mathbf{T}\left(\begin{pmatrix} x \\ y \end{pmatrix}\right)=\mathbf{A}\begin{pmatrix} x \\ y \end{pmatrix}+\mathbf{B}$ is only a linear transformation if $\mathbf{B}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ (the zero matrix). (Hint: you could start off by proving that $\mathbf{T}\begin{pmatrix} 0 \\ 0 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}$.)

Determinants

The Determinant of a square matrix $\mathbf{X}$ is written $|\mathbf{X}|$ or $\det \mathbf{X}$ and for a 2×2 matrix it is: $\begin{vmatrix} a & c \\ b & d \end{vmatrix} = ad - bc$. For 3×3 matrices, $\begin{vmatrix} a & b & c \\ e & f & g \\ h & i & j \end{vmatrix} = a\begin{vmatrix} f & g \\ i & j \end{vmatrix} - b\begin{vmatrix} e & g \\ g & j \end{vmatrix} + c\begin{vmatrix} e & f \\ h & i \end{vmatrix}$.

Determinants are defined for any $n\times n$ matrix in a similar way: for any $n\times n$ matrix,

$$\begin{vmatrix} x_{11} & x_{12} & x_{13} & \cdots & x_{1n} \\ x_{21} & x_{22} & x_{23} & \cdots & x_{2n} \\ x_{31} & x_{32} & x_{33} & \cdots & x_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & x_{n3} & \cdots & x_{nn} \end{vmatrix} = x_{11}\begin{vmatrix} x_{22} & x_{23} & \cdots & x_{2n} \\ x_{32} & x_{33} & \cdots & x_{3n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n2} & x_{n3} & \cdots & x_{nn} \end{vmatrix} \cdots$$

To get the first term you take the first element in the first row and multiply it by the determinant of the matrix after the first row and first column have been removed.

$$\begin{vmatrix} x_{11} & x_{12} & x_{13} & \cdots & x_{1n} \\ x_{21} & x_{22} & x_{23} & \cdots & x_{2n} \\ x_{31} & x_{32} & x_{33} & \cdots & x_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & x_{n3} & \cdots & x_{nn} \end{vmatrix} = x_{11}\begin{vmatrix} x_{22} & x_{23} & \cdots & x_{2n} \\ x_{32} & x_{33} & \cdots & x_{3n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n2} & x_{n3} & \cdots & x_{nn} \end{vmatrix} - x_{12}\begin{vmatrix} x_{21} & x_{23} & \cdots & x_{2n} \\ x_{31} & x_{33} & \cdots & x_{3n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n3} & \cdots & x_{nn} \end{vmatrix} \cdots$$

To get the second term you do the same sort of thing: you take the second element in the first row and then multiply it by the determinant where you cross out the row and column that it's in, but the second term is negative.

$$\begin{vmatrix} x_{11} & x_{12} & x_{13} & \cdots & x_{1n} \\ x_{21} & x_{22} & x_{23} & \cdots & x_{2n} \\ x_{31} & x_{32} & x_{33} & \cdots & x_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & x_{n3} & \cdots & x_{nn} \end{vmatrix} = x_{11}\begin{vmatrix} x_{22} & x_{23} & \cdots & x_{2n} \\ x_{32} & x_{33} & \cdots & x_{3n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n2} & x_{n3} & \cdots & x_{nn} \end{vmatrix} - x_{12}\begin{vmatrix} x_{21} & x_{23} & \cdots & x_{2n} \\ x_{31} & x_{33} & \cdots & x_{3n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n3} & \cdots & x_{nn} \end{vmatrix} + x_{13}\begin{vmatrix} x_{21} & x_{22} & \cdots & x_{2n} \\ x_{31} & x_{32} & \cdots & x_{3n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{vmatrix} \cdots$$

And repeat for every element along the top row, alternating between negative and positive terms.

Once this is completed you have a sum that contains $(n-1)\times(n-1)$ determinants so you can just repeat this until you get to a (very long) sum of 2×2 determinants which can simply be worked out. However, this method is very laborious and it is easy to make a mistake so we'll meet one later which is much better.

Determinants have many remarkable qualities, a few of which are:

1. If the matrix is transposed (ie. reflected along is diagonal, eg. $\begin{pmatrix} 3 & 2 & 1 \\ 4 & 3 & 2 \\ 5 & 4 & 3 \end{pmatrix} \to \begin{pmatrix} 3 & 4 & 5 \\ 2 & 3 & 4 \\ 1 & 2 & 3 \end{pmatrix}$) the determinant is unchanged.
2. Swapping two columns (rows) multiplies the determinant by –1.
3. Multiplying any column (row) by λ multiplies the determinant by λ (@).
4. Adding any column (row) to another column (row) leaves the determinant unchanged (@).
5. The determinant of any unit matrix is 1(@).

There are many others (especially one you'll find yourself later) but these are the basics. In fact, the determinant is the *only* function possible that fulfils the properties marked with @'s.

The Exercise

Each of these determinants has been changed (without changing its value) using a property above for each arrow. Try and find the numbers that go in each determinant and find the property used to change it and write it in the arrow. Some of the numbers in the determinants have been given as clues.

$$\begin{vmatrix} 1 & 3 & 2 \\ 1 & 2 & 1 \\ 4 & 0 & 3 \end{vmatrix} \Longrightarrow \begin{vmatrix} \; & \; & \; \\ 3 & 2 & \; \\ 2 & \; & \; \end{vmatrix} \Longrightarrow \begin{vmatrix} \; & \; & 7 \\ \; & \; & \; \\ \; & \; & \; \end{vmatrix}$$

Now, each arrow represents two changes using one of the properties above.

$$\begin{vmatrix} 1 & 6 & 0 & 3 \\ 1 & 0 & -1 & 0 \\ 2 & 2 & 0 & 2 \\ 3 & 0 & 4 & 5 \end{vmatrix} \Longrightarrow \begin{vmatrix} \; & 3 & \; & \; \\ \; & \; & -1 & \; \\ \; & 1 & \; & 4 \\ \; & \; & \; & \; \end{vmatrix} \Longrightarrow \begin{vmatrix} 3 & 0 & 4 & 10 \\ \; & \; & \; & \; \\ \; & \; & \; & \; \\ 1 & 3 & 0 & 6 \end{vmatrix}$$

Now, each arrow only represents one change again but each property may only be used once:

$$\begin{vmatrix} 1 & 2 & 0 \\ 3 & 4 & 1 \\ 0 & 5 & 2 \end{vmatrix} \Longrightarrow \begin{vmatrix} \; & \; & 2 \\ \; & \; & \; \\ \; & \; & 2 \end{vmatrix} \Longrightarrow - \begin{vmatrix} \; & 7 & \; \\ \; & \; & 3 \\ \; & 5 & \; \end{vmatrix} \Longrightarrow \begin{vmatrix} \; & \; & \; \\ \; & \; & \; \\ \; & \; & \; \end{vmatrix}$$

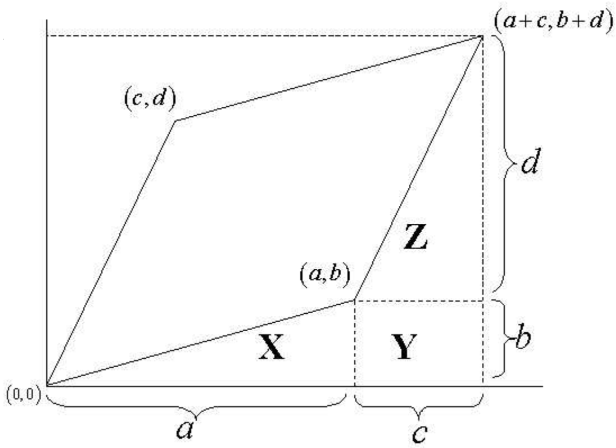
Determinants and Linear Transformations

You might be wondering why we would ever be interested in determinants (especially when they can be so laborious to calculate!) but there are many very good reasons, we'll focus on one…

Say a space is transformed by the matrix $\mathbf{T}=\begin{pmatrix} a & c \\ b & d \end{pmatrix}$ then $\mathbf{T}\begin{pmatrix} 0 \\ 0 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}$,

$\mathbf{T}\begin{pmatrix} 1 \\ 0 \end{pmatrix}=\begin{pmatrix} a \\ b \end{pmatrix}, \mathbf{T}\begin{pmatrix} 0 \\ 1 \end{pmatrix}=\begin{pmatrix} c \\ d \end{pmatrix}$, and $\mathbf{T}\begin{pmatrix} 1 \\ 1 \end{pmatrix}=\begin{pmatrix} a+c \\ b+d \end{pmatrix}$; hence the unit square becomes…

This shape can be split into four equal sections (as shown; the area of one is labelled X). Then the area of the whole is



We can find the area of the shape by finding the areas of $\mathbf{X},\mathbf{Y},\mathbf{Z}$ and taking twice their sum from the area of the large square $\mathbf{W}$ (that meets the shape at the bottom-left and top-right):

$$\mathbf{W}-2(\mathbf{X}+\mathbf{Y}+\mathbf{Z})$$
$$=(a+c)(b+d)-2\left(\frac{1}{2}ab+bc+\frac{1}{2}cd\right)$$
$$=(ab+ad+bc+cd)-(ab+2bc+cd)$$
$$=ad-bc$$

Notice that $\begin{vmatrix} a & c \\ b & d \end{vmatrix}=ad-bc$, ie. the determinant of a transformation matrix is the area of a transformed unit square. Or rather, more generally, (since the area of the unit square is 1) the determinant is the ratio through which every shape's area is multiplied when it is transformed.

More impressively, this is not just true for 2-dimensions: given 3D (or 4D or 5D or nD) spaces this still works (but the proof of this is a little more involved); ie. in 3D, the determinant gives the volume of the unit cube after transformation.

Another Exercise

1. For each of the transformed unit squares you have drawn above, try and calculate their area both using determinants and not.
2. *With your answer to 2(c) (of Linear Transformations) above in mind, why is it true that $|\mathbf{AB}|=|\mathbf{A}||\mathbf{B}|$.
3. *The inverse of a matrix $\mathbf{X}$ is a matrix $\mathbf{X}^{-1}$ such that $\mathbf{X}^{-1}\mathbf{X}$ is the identity matrix. Using (2), when can a matrix not be inverted (ie. when is it impossible to find an inverse).

4. *What happens to 2 dimensional space transformed by a matrix $\mathbf{T}$ if $|\mathbf{T}| = 0$?
(There are two possibilities. Now, can you go further and find the n possibilities for an n dimensional space?)

Pivotal Condensation

When using large matrices the determinant can be very difficult to find because the process is long and it is very easy to make a mistake (not to mention the waste of paper). So, there is a quicker way to do it: pivotal condensation. For an example:

$$\begin{vmatrix} 1 & 2 & 3 & 0 \\ 0 & 3 & 1 & 2 \\ 1 & 0 & 1 & 1 \\ 1 & 2 & 3 & 0 \end{vmatrix} = \begin{vmatrix} \|1\| & [2 & 3 & 0] \\ [0] & 3 & 1 & 2 \\ [1] & 0 & 1 & 1 \\ [1] & 2 & 3 & 0 \end{vmatrix}$$

Firstly, a pivot is chosen (in this case the top left element) and its row and column are singled out.

$$\begin{vmatrix} \|1\| & [2 & 3 & 0] \\ [0] & 1\times3-2\times0 & 1\times1-3\times0 & 1\times2-0\times0 \\ [1] & 1\times0-2\times1 & 1\times1-3\times1 & 1\times1-0\times1 \\ [1] & 1\times3-2\times1 & 1\times3-3\times1 & 1\times0-0\times1 \end{vmatrix} = \frac{1}{1}\begin{vmatrix} 3 & 1 & 2 \\ -2 & -2 & 1 \\ 1 & 0 & 0 \end{vmatrix}$$

If for an element *x*, the pivot is *l*, the singled-out element in the same row (column) is *r* (*c*), then that element is replaced with *lx*–*rc*.

Finally, the singled out columns are removed and the determinant is divided by the pivot to the power N – 2, where the matrix is N by N, then multiplied by $(-1)^{m+n}$ where *m* and *n* are the 'coordinates' of the pivot in the matrix (the top left being *m*=*n*=1). If the determinant isn't yet small enough (ie. two by two) then just repeat the process. Obviously best to choose a 1 as pivot.

The Exercise

1. To show how useful pivotal condensation is, try using it to find the following using the marked pivot for the first time:
   - $\begin{vmatrix} \boxed{1} & -7 & 0 \\ 2 & 3 & 1 \\ 4 & 2 & 2 \end{vmatrix}, \begin{vmatrix} 0 & 2 & 2 \\ 2 & 1 & \boxed{1} \\ 2 & 3 & 1 \end{vmatrix}$
   - $\begin{vmatrix} 2 & 2 & 3 & 2 & 1 \\ 2 & 0 & 1 & 3 & 2 \\ 1 & 1 & \boxed{2} & 0 & 1 \\ 3 & 2 & 3 & 1 & 2 \\ 1 & 0 & 1 & 2 & 0 \end{vmatrix}$
2. *Why does this work?