EXPECTED VALUES – APPENDIX – SOME SERIES

Binomial series : for $(1 + x)^n$ where n is an integer ≥ 0

n=0 $(1 + x)^0 = 1$
n=1 $(1 + x)^1 = 1 + x$
n=2 multiply above by $(1 + x)$
$(1 + x)^2 = (1 + x)(1 + x)$
$= 1 + x + x + x^2 = 1 + 2x + x^2$
n=3 multiply above by $(1 + x)$
$(1 + x)^3 = (1 + x)(1 + 2x + x^2)$
$= 1 + 2x + x^2 + x + 2x^2 + x^3 = 1 + 3x + 3x^2 + x^3$
etc ……

Geometric series : $(1 – x)^{-1}$, where $–1 < x < 1$

Initially for any x, set
$S_r = 1 + x + x^2 + x^3 + …… + x^r$ where r is an integer ≥ 0. Refer to this equation as (1).
Multiply above by x
$xS_r = x + x^2 + x^3 + …… + x^r + x^{r+1}$. Refer to this equation as (2).
Subtract (2) from (1), giving
$S_r – xS_r = 1- x^{r+1}$,
ie $(1 – x)S_r = 1 - x^{r+1}$

Hence $S_r = \dfrac{(1-x^{r+1})}{(1-x)}$.

For $–1 < x < 1$, let r tend to infinity and S_r tend to S.

Thus $S = \dfrac{1}{(1-x)} = (1 – x)^{-1}$

So, when $–1 < x < 1$, we have $(1 – x)^{-1} = 1 + x + x^2 + x^3 + ……$

Negative binomial series : for $(1 – x)^m$ where $–1 < x < 1$ and m is an integer < 0

m= -1 $(1 – x)^{-1} = 1 + x + x^2 + x^3 + ……$ where $–1 < x < 1$
multiply above by $(1 – x)^{-1} = 1 + x + x^2 + x^3 + ……$
m= -2 $(1 – x)^{-1}(1 – x)^{-1} = (1 + x + x^2 + x^3 + ……)(1 + x + x^2 + x^3 + ……)$
$= 1 + x + x^2 + x^3 + ……$
$x + x^2 + x^3 + ……$
$x^2 + x^3 + ……$
$x^3 + ……$
$+ ……$
So $(1 – x)^{-2} = 1 + 2x + 3x^2 +4 x^3 + ……$ where $–1 < x < 1$
By multiplying above $(1 – x)^{-1} = 1 + x + x^2 + x^3 + ……$ we can similarly derive
m= -3 $(1 – x)^{-3} = 1 + 3x + 6x^2 + 10 x^3 + ……$ where $–1 < x < 1$

Pascal's Triangle for coefficients in binomial series

With appropriate interpretation the coefficients of powers of x in the binomial expansions for $(1 + x)^n$ where n is an integer ≥ 0 can be read horizontally and those for $(1 – x)^m$ where m is an integer < 0 and $–1 < x < 1$ can be read downhill diagonally.

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1

Entries in further rows can be computed as the sum of the two adjacent entries immediately above.