EXPECTED VALUES

Introduction to probability **Discrete variable (or "count" variable)**

A statistical experiment or trial is a procedure which yields one (un-predetermined) result out of a set of known possible results.

This set of known possible results ("possibilities") is called the results space, or sometimes the sample space, of the experiment.

e.g.
(i) Number of heads X obtained on a single toss of a coin.
Results space X = 0, 1.
(ii) Score X obtained on a single throw of a standard die
Results space X = 1, 2, 3, 4, 5, 6.
(iii) Number of children in a randomly selected family
Results space X = 0, 1, 2, 3, …….

Definitions: If to each result x_i in a results space we associate a fraction p_i (lying between 0 and 1 inclusive) such that the sum of the p_i's is 1, then we say the p_i's define a probability distribution on the results space.

p_i is referred to as "the probability of x_i occurring" or "the probability of x_i".
A variety of notations can be used as appropriate: $\Pr(X = x_i) = p_i$, $\Pr(x_i) = p_i$ etc.

There is a well-established convention in which in relation to the results space we use upper-case (capital) letters, eg X, to represent strings of words and lower-case letters to represent values,

Examples
1) X is the number of heads obtained on a single toss of a fair coin.

Results space	X:	0	1
Probabilities		p_0	p_1

$p_0 + p_1 = 1$ from definition of probability distribution
$p_0 = p_1$ since the given coin is a fair coin
$\therefore$ $p_0 = ½$, $p_1 = ½$.

2) Similarly for : X is the score on a single throw of a fair die.

Results space	X:	1	2	3	4	5	6
Probabilities		p_1	p_2	p_3	p_4	p_5	p_6

Here $p_1 = {}^1/_6$, etc.

Probability theory goes on to encompass many common-sense concepts and useful theorems. However, except for the idea of an expected value, we will not be exploring any of these concepts or theorems ibn this session.
.

Expected values **Discrete variable**

Given the probability distribution

Results space	X	x_1	x_2	x_3	……
Probabilities		p_1	p_2	p_3	……

Suppose that we take a sample of n observations and the corresponding frequencies, ie numbers of times observed, are

Results space	X	x_1	x_2	x_3	……
Frequencies		f_1	f_2	f_3	……

Equivalently we have the proportions

Results space	X	x_1	x_2	x_3	……
Proportions		$\frac{f_1}{n}$	$\frac{f_2}{n}$	$\frac{f_3}{n}$	……

We have the intuitive notion that, "in the long run" (ie as $n \to \infty$), $\frac{f_1}{n} \to p_1$ etc.

The sample mean $\bar{x}$ is given by $\bar{x} = \dfrac{x_1 f_1 + x_2 f_2 + x_3 f_3 + ...}{n}$

This can be re-written as $\bar{x} = x_1 \frac{f_1}{n} + x_2 \frac{f_2}{n} + x_3 \frac{f_3}{n} + ...$

"In the long run" as $n \to \infty$ the right hand side $\to$ $x_1p_1 + x_2p_2 + x_3p_3 + ...$

This then is the intuitive expected "long run" mean value, or briefly, the <u>expected value</u> of the given distribution of X. Hence:

<u>Definition</u>
Given the probability distribution

Results space	X	x_1	x_2	x_3	……
Probabilities		p_1	p_2	p_3	……

The quantity $x_1p_1 + x_2p_2 + x_3p_3 + ... = \Sigma_i x_i p_i$ is called the <u>expected value</u> (or expectation or mathematical expectation) of X and is denoted by E(X) (or E(X)).
So: $E(X) = \Sigma_i x_i p_i$. (provided, of course, that this sum exists, ie is finite)

The definition and notation can be immediately extended to functions of X
eg $E(X^2) = \Sigma_i x_i^2 p_i$
$E(2X + 1) = \Sigma_i (2x_i + 1)p_i$
Note: this can be re-written as $\Sigma_i 2x_ip_i + \Sigma_i p_i = 2\, \Sigma_i x_i p_i + 1 = 2E(X) + 1$
- this is an example of a general result coming up shortly.

$E(\varphi(X)) = \Sigma_i\, \varphi(x_i)p_i$

Basically, in practice, take whatever is inside the brackets, work out its value for each value in the results space, multiply by the corresponding probability, and add.

Examples

1. X is the score on single throw of fair die

Results space	X	1	2	3	4	5	6
Probabilities		$^1/_6$	$^1/_6$	$^1/_6$	$^1/_6$	$^1/_6$	$^1/_6$

$E(X) = 1\times{}^1/_6 + 2\times{}^1/_6 + 3\times{}^1/_6 + 4\times{}^1/_6 + 5\times{}^1/_6 + 6\times{}^1/_6 = 3\tfrac{1}{2}$
$E(X^2) = 1^2\times{}^1/_6 + 2^2\times{}^1/_6 + 3^2\times{}^1/_6 + 4^2\times{}^1/_6 + 5^2\times{}^1/_6 + 6^2\times{}^1/_6 = 15{}^1/_6$

Note that in general $E(X^2) \neq (E(X))^2$
$E(\varphi(X)) \neq \varphi(E(X))$

2, X is the number of heads on a single toss of a fair coin

Results space	X	0	1
Probabilities		$^1/_2$	$^1/_2$

By similar calculation to above $E(X) = \tfrac{1}{2}$

Comment
Often expectation is the first theoretic concept met in the study of statistics/probability.
It is frequently easier to prove an expectation result (theorem) in general and apply it to a particular problem than it is to derive the result from first principles each and every time that it is required.,

Expected values Discrete variable Exercises

1. X is the number of heads obtained on a single toss of a fair coin.

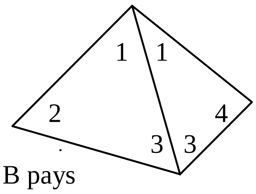
   Calculate (i) E(X) (ii) $E(X^2)$ (iii) $E(\frac{1}{X})$ (iv) E(2X) (v) E(X+3) (vi) E(3X+2)

2. X is score obtained on single throw of a fair standard die.

   Calculate (i) $E(\frac{1}{X})$ (ii) E(X!) (iii) E(sum of possible scores $\leq$ X)

   For (ii) you will need to recall that for positive integer x we have
   $x! = x \times (x-1) \times (x-2) \times \ldots \times 1$

3. ‘A’ has a fair standard six-faced die.
   ‘B’ has a fair four-faced pyramid die as shown.
   ‘A’ observes that the total of the possible scores on his die is 21 points and that on B’s is 10 points.
   He reckons that if they throw their dice together and he pays B 2p for each point he (B) scores, while B pays him 1p for each point he (A) scores, then he (A) can expect to make a small profit if the dice are thrown many times.
   Is he right?



4. Given the probability distribution

Results space	X	x_1	x_2	x_3	……
Probabilities		p_1	p_2	p_3	……

   Prove that for constants k, c   (i) E(X+k)=E(X)+k   (ii) E(cX) = cE(X)

5. (i) If the faces of a fair die are marked 11, 12, 13, 14, 15, 16 what will be the expected value of the distribution of scores of a single throw of this die?
   (ii) If the faces of a fair die are marked 6, 12, 18, 24, 30, 36 what will be the expected value of the distribution of scores of a single throw of this die?

Possible answers.

1. (i) ½ (ii) ½ (iii) does not exist (iv) 1 (v) 3½ (vi) 3½
2. (i) $^{147}/360$ (ii) 145½ (iii) 9⅓
3. No

5 (i) 13½ (ii) 21

Expected values Coin tossing run lengths Exercise

A fair coin us tossed a large number of times. X is a run length, that is the count of a continuous sequence of H (or T) from the first occurrence to the last before the next T (or H) occurs.

Results Space	X	1	2	3	4	…….
Probabilities		½	$(½)^2$	$(½)^3$	$(½)^4$	……

Note
The sum of the probabilities is

$$½ + (½)^2 + (½)^3 + (½)^4 + \ldots\ldots = ½\,(1 + ½ + (½)^2 + (½)^3 + \ldots) = ½ \times (1 - ½)^{-1} = 1$$

as required.

Work out the value of E(X).