University of Liverpool Maths Club

Ringing the Changes

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English church bells do not ring tunes. Instead they ring *changes*, and there is lots of interesting mathematics to be found here.

Before we explain what change ringing is and start to look at the maths involved, we'll quickly look at why it's not possible to ring tunes on church bells.

To do this, we need to know how a church bell is rung.

A church bell is quite large – the smallest ones, like those at Rainhill, weigh about the same as me. The biggest change ringing bell at Liverpool Cathedral weighs just over 4 tonnes and is the heaviest in the world.

The bell is attached to a wheel and a rope. When it is rung, the bell starts off balanced mouth-upwards. When the rope is pulled, the bell falls off the balance and turns through a whole circle before stopping mouth-upwards again. Next time the rope is pulled, it goes back again. Each time it takes about two seconds from starting to pull the rope before the bell sounds.

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Each ringer can change the speed of his or her bell just a little bit, enough to move one place earlier or one place later in the sequence. In this way we can move the bells into different orders. This is what change ringing is about!

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Remember that each bell is only allowed to move one place from where it started.

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1 2 3 4 5 6 7 8

Next time, bell number 1 is going to ring a little bit more slowly and move into 2nd place. Who is going to move into first place?

Remember that each bell is only allowed to move one place from where it started. So it must be bell number 2 which moves into first place, by ringing a little more quickly.

2 1 3 4 5 6 7 8

Bells 1 and 2 have swapped places. This is an example of a *change*.

In this example the bells changed from one order

$$1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8$$

to another order

$$2 \quad 1 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8$$

by some bells ringing slightly more quickly or slowly. These orders are called *permutations* by mathematicians and *rows* by bell ringers. The object of change ringing is to ring sequences of rows without repeating any.

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How many different rows are there on three bells? On four bells?

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These rules are what make the interesting mathematics of change ringing.

Let's put this into practice with a little exercise.

In the exercise we just did, the first change involved all the bells changing place in pairs.

1 2 3 4 5 6 7 8

X X X X

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$$
\begin{array}{ccccccccccccccc}
1 & & 2 & & 3 & & 4 & & 5 & & 6 & & 7 & & 8 \\
 & \times & & & & \times & & & & \times & & & & \times & \\
2 & & 1 & & 4 & & 3 & & 6 & & 5 & & 8 & & 7
\end{array}
$$

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$$\begin{array}{ccccccccc}
 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
\text{X} & \multicolumn{2}{c}{\times} & \multicolumn{2}{c}{\times} & \multicolumn{2}{c}{\times} & \multicolumn{2}{c}{\times} \\
 & 2 & 1 & 4 & 3 & 6 & 5 & 8 & 7
\end{array}$$

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$$\begin{array}{ccccccccc} & 2 & 1 & 4 & 3 & 6 & 5 & 8 & 7 \\ \mathrm{X} & \multicolumn{2}{c}{\times} & \multicolumn{2}{c}{\times} & \multicolumn{2}{c}{\times} & \multicolumn{2}{c}{\times} \end{array}$$

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All the bells end up back where they started!

Since any change involves bells swapping in pairs, this is true for any change: doing it twice consecutively takes you back to the row you started with.

Instead, the second time we did a different change, where the bells in positions 1 and 8 stay still and the remaining ones swap in pairs:

$$\begin{array}{cccccccccccccccc}
 & 1 & & 2 & & 3 & & 4 & & 5 & & 6 & & 7 & & 8 \\
\mathrm{X} & & \times & & & & \times & & & & \times & & & & \times & \\
 & 2 & & 1 & & 4 & & 3 & & 6 & & 5 & & 8 & & 7 \\
 & | & & & \times & & & & \times & & & & \times & & & |
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\times & & \times & & & & \times & & & & \times & & & & \times & \\
 & 2 & & 1 & & 4 & & 3 & & 6 & & 5 & & 8 & & 7 \\
 & | & & & \times & & & & \times & & & & \times & & & | \\
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 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
\mathrm{X} & \multicolumn{2}{c}{\times} & \multicolumn{2}{c}{\times} & \multicolumn{2}{c}{\times} & \multicolumn{2}{c}{\times} \\
 & 2 & 1 & 4 & 3 & 6 & 5 & 8 & 7 \\
18 & | & \multicolumn{2}{c}{\times} & \multicolumn{2}{c}{\times} & \multicolumn{2}{c}{\times} & | \\
 & 2 & 4 & 1 & 6 & 3 & 8 & 5 & 7
\end{array}$$

Bell ringers write this change as ‘18’, because the bells in positions 1 and 8 stay still. Now we can do change ‘X’ again.

If we carry on alternating the changes ‘X’ and ‘18’, we get back to rounds after 16 changes.

	1	2	3	4	5	6	7	8
X								
	2	1	4	3	6	5	8	7
18								
	2	4	1	6	3	8	5	7
X								
	4	2	6	1	8	3	7	5
18								
	4	6	2	8	1	7	3	5
X								
	6	4	8	2	7	1	5	3
18								
	6	8	4	7	2	5	1	3
X								
	8	6	7	4	5	2	3	1
18								
	8	7	6	5	4	3	2	1
X								
	7	8	5	6	3	4	1	2
18								
	7	5	8	3	6	1	4	2
X								
	5	7	3	8	1	6	2	4
18								
	5	3	7	1	8	2	6	4
X								
	3	5	1	7	2	8	4	6
18								
	3	1	5	2	7	4	8	6
X								
	1	3	2	5	4	7	6	8
18								
	1	2	3	4	5	6	7	8

Notice what we've done.

- Doing two 'X' changes or two '18' changes consecutively takes us back to the same row. So to get new rows we must alternate them – do a 'X' change, then a '18' change, then a 'X' change, and so on.

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- Even if we alternate 'X' changes and '18' changes, we get back to rounds after 16 changes.

That means that the rows on the previous slide are all the ones we can get to using the two changes 'X' and '18'. In mathematical terms, that means that they form a *subgroup* of the group of all permutations on 8 bells. We will come back to this idea later.

Let us think for a moment about ringing on three bells. Here there are six rows, and only two possible changes. Starting from rounds, the first and second bells can swap and the third stay still:

$$\begin{array}{ccc} 1 & & 2 & & 3 \\ & \times & & & | \end{array}$$

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$$\begin{array}{ccc} 1 & 2 & 3 \\ \multicolumn{2}{c}{\times} & | \\ 2 & 1 & 3 \end{array}$$

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$$\begin{array}{ccc} 1 & 2 & 3 \\ \multicolumn{2}{c}{\times} & | \\ 2 & 1 & 3 \end{array}$$

or the first bell can stay still, and the second and third bells swap.

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$$\begin{array}{ccc} 1 & 2 & 3 \\ | & \multicolumn{2}{c}{\times} \\ 1 & 3 & 2 \end{array}$$

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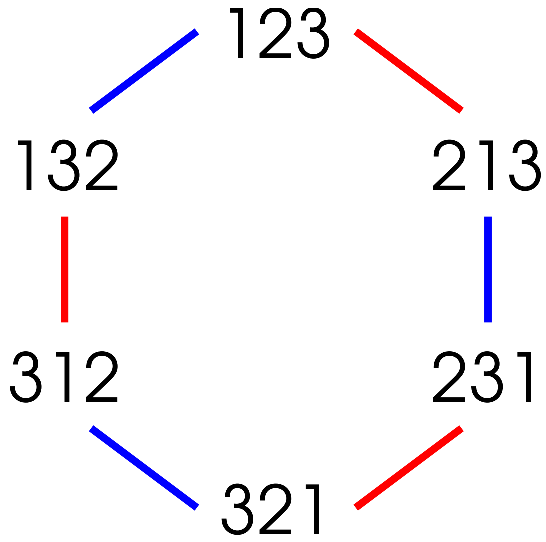
$$\begin{array}{cccccc} & 1 & & 2 & & 3 \\ 3 & & \times & & & | \\ & 2 & & 1 & & 3 \end{array}$$

or the first bell can stay still, and the second and third bells swap.

$$\begin{array}{cccccc} & 1 & & 2 & & 3 \\ 1 & | & & & \times & \\ & 1 & & 3 & & 2 \end{array}$$

These changes are called ‘3’ and ‘1’ respectively.

We can draw a diagram, or *graph*, to show all the ways of getting from one row to another on three bells. Each corner represents a row, and the lines between them represent the changes which go from one row to another. The lines are coloured blue for the change ‘1’ and red for the change ‘3’.



The graph shows us that there are just two ways of visiting all the rows exactly once.

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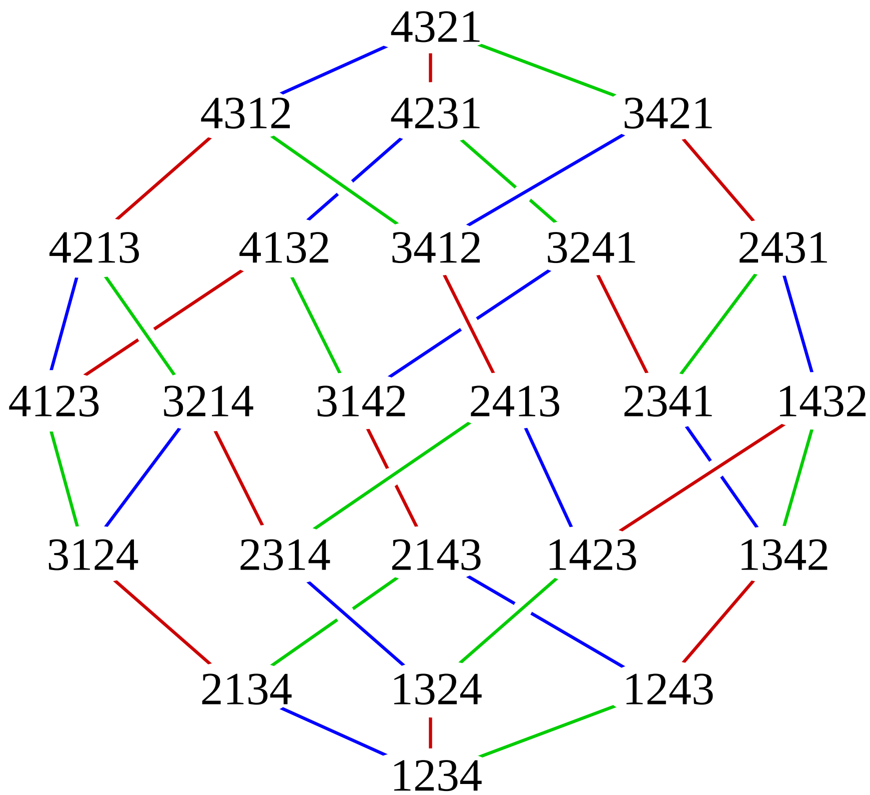
	1	2	3
3	2	1	3
1	2	3	1
3	3	2	1
1	3	1	2
3	1	3	2
1	1	2	3

	1	2	3
1	1	3	2
3	3	1	2
1	3	2	1
3	2	3	1
1	2	1	3
3	1	2	3

On four bells there are 24 rows and 4 different changes: 'X', '14', '12' and '34'.



One nice way to visualise the graph for four bells is to draw it on a truncated octahedron. It looks a lot more complicated than for three bells, but we can still spot some ways of getting to all the rows exactly once.

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On five bells, things get even more complicated!

To finish, we'll go back to the series of changes on eight bells that we rang earlier.

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	1	2	3	4	5	6	7	8
×								
	2	1	4	3	6	5	8	7
18								
	2	4	1	6	3	8	5	7
×								
	4	2	6	1	8	3	7	5
18								
	4	6	2	8	1	7	3	5
×								
	6	4	8	2	7	1	5	3
18								
	6	8	4	7	2	5	1	3
×								
	8	6	7	4	5	2	3	1
18								
	8	7	6	5	4	3	2	1
×								
	7	8	5	6	3	4	1	2
18								
	7	5	8	3	6	1	4	2
×								
	5	7	3	8	1	6	2	4
18								
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×								
	3	5	1	7	2	8	4	6
18								
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×								
	1	3	2	5	4	7	6	8
18								
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What happens if we ring the same changes but, instead of starting from rounds, we start from a different row? We might get this.

1 3 5 2 7 4 8 6

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X	1	3	5	2	7	4	8	6
	3	1	2	5	4	7	6	8

What happens if we ring the same changes but, instead of starting from rounds, we start from a different row? We might get this.

$$\begin{array}{r|cccccccc} & 1 & 3 & 5 & 2 & 7 & 4 & 8 & 6 \\ \times & 3 & 1 & 2 & 5 & 4 & 7 & 6 & 8 \\ 18 & 3 & 2 & 1 & 4 & 5 & 6 & 7 & 8 \end{array}$$

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	1	3	5	2	7	4	8	6
X								
	3	1	2	5	4	7	6	8
18								
	3	2	1	4	5	6	7	8
X								
	2	3	4	1	6	5	8	7
18								
	2	4	3	6	1	8	5	7
X								
	4	2	6	3	8	1	7	5
18								
	4	6	2	8	3	7	1	5
X								
	6	4	8	2	7	3	5	1
18								
	6	8	4	7	2	5	3	1
X								
	8	6	7	4	5	2	1	3
18								
	8	7	6	5	4	1	2	3
X								
	7	8	5	6	1	4	3	2
18								
	7	5	8	1	6	3	4	2
X								
	5	7	1	8	3	6	2	4
18								
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	1	5	3	7	2	8	4	6
18								
	1	3	5	2	7	4	8	6

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In exactly the same way, we can get from any row in the second list to any other row in the second list using the changes ‘X’ and ‘18’.

Notice that the row we started with the second time

$$1 \quad 3 \quad 5 \quad 2 \quad 7 \quad 4 \quad 8 \quad 6$$

wasn't one of the rows in the original list: in other words, we can't get to it from rounds using the changes 'X' and '18'. We will call this row R.

We can deduce something very useful. *None* of the rows in the second list appears in the first list! How do we know that without checking them all, one by one? We will prove it.

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This will be a *proof by contradiction*. We will suppose that there *is* a row which appears in both the first list and the second list, and deduce something untrue; that will mean that our supposition cannot be true.

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- We could get from the row S to the row R using changes 'X' and '18' (because S is in the second list).

- We could get from rounds to the row S using changes 'X' and '18' (because S is in the first list).

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- We could get from the row S to the row R using changes 'X' and '18' (because S is in the second list).

- We could get from rounds to the row S using changes 'X' and '18' (because S is in the first list).

Therefore we could get from rounds to the row R using changes 'X' and '18', and so R would have to be in the first list.

But R isn't in the first list. That means that our supposition was wrong: there can be no such S. No row from the second list can appear in the first list, and we have finished the proof.

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Remember that the rows in the first list are mathematically called a *subgroup* of the group of permutations of 8 bells. When we did the same procedure but starting from a different row, we formed what is called a *coset* of that subgroup in our second list of rows. We have really been proving a very general mathematical result about groups: that two different cosets of a subgroup do not overlap.