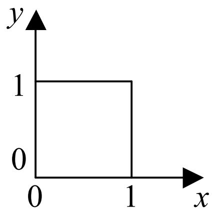
Maths Club 29th January 2005

Transforming spaces, transforming shapes

We shall concentrate on two-dimensional spaces, namely planes.

On our starter plane we will lay an orthogonal Cartesian co-ordinate grid with x, y axes. Every point on the plane is uniquely identified by its (x,y) co-ordinates.

In each example we will specify a particular region of our starter plane. The region will be identified typically in terms of inequalities relative to its "boundary segments" where these boundary segments are expressed in terms of relationships of (x,y) co-ordinate values and/or constants.

eg 

The square shown is specified as the set of points having (x,y) co-ordinates such that $0 \le x \le 1$ and $0 \le y \le 1$.
There are four boundary segments. Starting at the origin and walking anticlockwise round the boundary we have successively $\{y = 0, x \text{ increases from } 0 \text{ to } 1\}$, $\{x = 1, y \text{ increases from } 0 \text{ to } 1\}$, $\{y = 1, x \text{ decreases from } 1 \text{ to } 0\}$, $\{x = 0, y \text{ decreases from } 1 \text{ to } 0\}$.

(Aside: The square $0 < x < 1, 0 < y < 1$ has the same boundary segments but technically they would be part of the outside of the square rather than be part of the square itself. In the context of what we will be doing, the distinction is not a problem.)

In some circumstances, but not in this example, it may be necessary to regard particular individual points (typically some corner points) as boundary segments as requiring their own individual consideration.

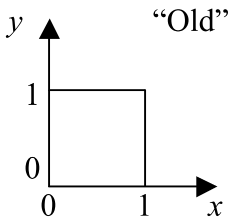
Transformation:

It is often the case that we may require to work not in terms of x and y directly but in terms of new variables which are some functions of x and y. For our purposes here we will confine our attention to cases where we have two such new variables which we will usually, but not exclusively, denote by u and v.

Visualise our starter plane being distorted / changed / transformed by this process and having a new orthogonal Cartesian co-ordinate grid with u, v axes laid upon it.

Here we are going to work out how the boundary segments of our given region are changed / transformed by the process.

Example 1



The square shown is the set of points (x,y) such that $0 \le x \le 1$ and $0 \le y \le 1$.

Consider the transformation of the boundary segments using:

$$u = x + y,\ \ v = x - y$$

Starting at the origin and "walking" anticlockwise round the boundary we have successively

$\{y = 0; x \text{ increases from } 0 \text{ to } 1\}$,

$\therefore \{u = x \text{ and } v = x \text{ so } v = u \text{ ; both increase from } 0 \text{ to } 1\}$

$\{x = 1; y \text{ increases from } 0 \text{ to } 1\}$,

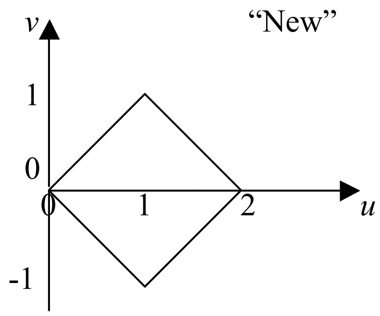
$\therefore \{u = 1 + y, v = 1 - y \text{ so } v = 2 - u \text{ ; } u \text{ increases from } u = 1 + 0 = 1 \text{ to } 1 + 1 = 2$
$\text{and } v \text{ decreases from } v = 1 - 0 = 1 \text{ to } 1 - 1 = 0\}$

$\{y = 1; x \text{ decreases from } 1 \text{ to zero}\}$,

$\therefore \{u = x + 1, v = x - 1 \text{ so } v = u - 2 \text{ ; } u \text{ decreases from } u = 1 + 1 = 2 \text{ to } 0 + 1 = 1,$
$\text{and } v \text{ decreases from } v = 1 - 1 = 0 \text{ to } 0 - 1 = -1\}$

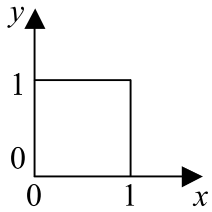
$\{x = 0; y \text{ decreases from } 1 \text{ to } 0\}$

$\therefore \{u = y, v = -y \text{ so } v = -u \text{ ; } u \text{ decreases from } 1 \text{ to } 0 \text{ and } v \text{ increases from } -1 \text{ to } 0\}$



Exercises

1

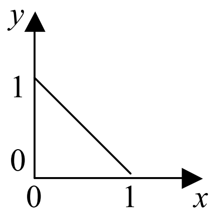


The square shown is the set of points (x,y) such that $0 < x < 1$ and $0 < y < 1$.

Consider the transformation of the boundary segments using:

$$u = x,\ \ v = x\,y$$

2

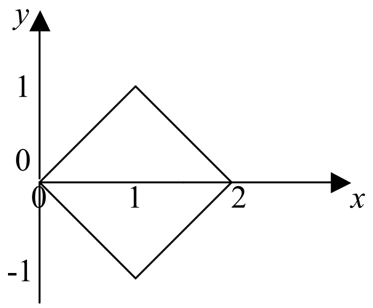


The triangle shown is the set of points (x,y) such that $0 \le x \le 1$ and $0 \le y \le 1 - x$.

Consider the transformation of the boundary segments using:

$$u = x^{1/2},\ \ v = y^{1/2}$$

3



The square shown is the set of points (x,y) such that

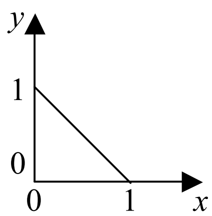
$\{-x \le y \le x$ and $0 \le x \le 1\}$ or

$\{-2 + x \le y \le 2 - x$ and $1 < x \le 2\}$

Consider the transformation of the boundary segments using:

$$u = \frac{1}{2}(x + y), v = \frac{1}{2}(x - y)$$

4

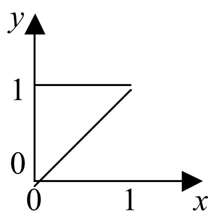


The triangle shown is the set of points (x,y) such that $0 \le x \le 1$ and $0 \le y \le 1 - x$.

Consider the transformation of the boundary segments using:

$$u = x,\ \ v = x + y$$

5

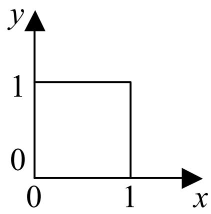


The triangle shown is the set of points (x,y) such that $0 \le x \le 1, \mathrm{x} \le y \le 1$.

Consider the transformation of the boundary segments using:

$$u = x, v = xy$$

6

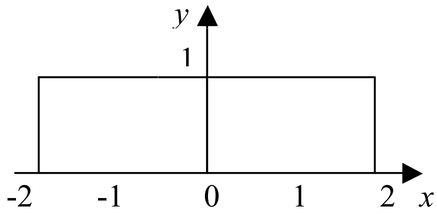


The square shown is the set of points (x,y) such that $0 \le x \le 1, 0 \le y \le 1$.

Consider the transformation of the boundary segments using:

$$u = x + y, v = x + y$$

7



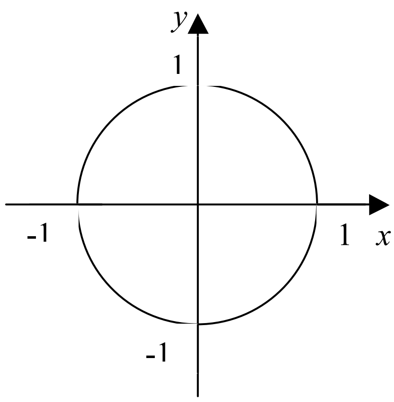
The rectangle shown is the set of points (x,y) such that $-2 \le x \le 2$ and $0 \le y \le 1$.

Consider the transformation of the boundary segments using:

$$u = x^2, \; v = y$$

What happens to the "old" y axis under this transformation?

8



The circle shown is the set of points (x,y) such that $x^2 + y^2 \le 1$

Consider the transformation of the boundary using:

$$u = \frac{1}{\sqrt{2}}(x + y), v = \frac{1}{\sqrt{2}}(x - y)$$

What happens to the "old" x axis under this transformation?

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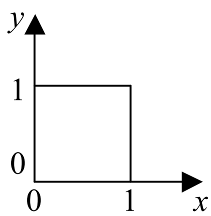
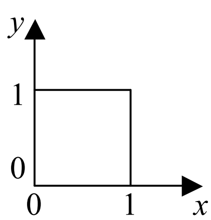


The square shown is the set of points (x,y) such that $0 \le x \le 1, 0 \le y \le 1$.

Consider the transformation of the boundary segments using:

$$u = x^{½}, \; v = y^{½}$$

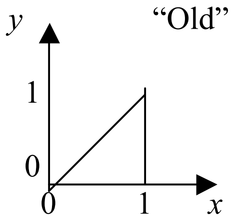
10



The square shown is the set of points (x,y) such that $0 \le x \le 1$ and $0 \le y \le 1$.

Consider the transformation of the boundary segments using:
$u = 2x + y, v = x + 2y$

Example 2



The triangle shown is the set of points (x,y) such that $0 \le x \le 1$ and $0 \le y \le x$.

Consider the transformation of the boundary segments using:

$$u = x, v = \frac{y}{x}$$

Starting at the origin and "walking" anticlockwise round the boundary we have successively

$\{y = 0;\ x \text{ increases from } 0 \text{ to } 1\}$,

$\therefore \{u = x \text{ and } v = 0;\ u \text{ increases from } 0 \text{ to } 1\}$

$\{x = 1;\ y \text{ increases from } 0 \text{ to } 1\}$,

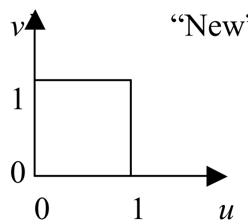
$\therefore \{u = 1, v = y;\ v \text{ increases from } 0 \text{ to } 1\}$

$\{y = x;\ x \text{ decreases from } 1 \text{ to zero}\}$,

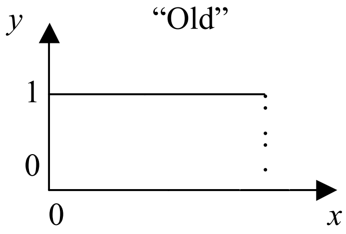
$\therefore \{v = 1;\ u \text{ decreases from } 1 \text{ to } 0\}$

$\{x = 0;\ y = 0\}$

$\therefore$ in limit $\{u = 0, v \text{ indeterminate but lies between } 1 \text{ and } 0\}$



Example 3



The region shown is the set of points (x,y) such that $0 < x < \infty$ and $0 < y < 1$.

Consider the transformation of the boundary segments using:

$$u = x, v = xy$$

Starting at the origin and "walking" anticlockwise round the boundary (regard the dotted line as a notional boundary segment) we have successively

$\{y = 0; x \text{ increases from } 0 \text{ to } \infty\}$,

$\therefore \{u = x \text{ and } v = 0; u \text{ increases from } 0 \text{ to } \infty\}$

$\{x \to \infty; y = 0\}$,

$\therefore \{u \to \infty, v \text{ indeterminate "between } 0 \text{ and } \infty\text{"}\}$

$\{x \to \infty, y \text{ increases from } 0 \text{ to } 1\}$

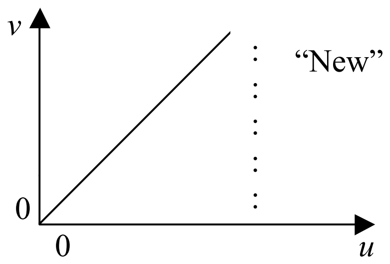
$\therefore \{u \to \infty, v \to \infty\}$

$\{y = 1; x \text{ decreases to zero}\}$,

$\therefore \{u = x, v = xy \text{ so } v = u; v = u, u \text{ decreases to } 0 \text{ and } v \text{ decreases to } 0\}$

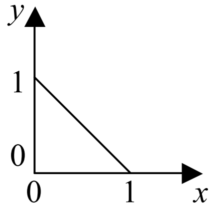
$\{x = 0; y \text{ decreases from } 1 \text{ to } 0\}$

$\therefore \{u = 0, v = xy = 0\}$



Exercises - continued

11



The triangle shown is the set of points (x,y) such that $0 < x < 1$ and $0 < y < 1 - x$.

Consider the transformation of the boundary segments using:

$$u = x, v = \frac{y}{1-x}$$

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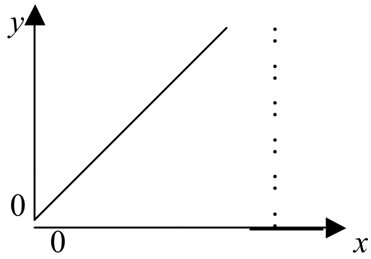
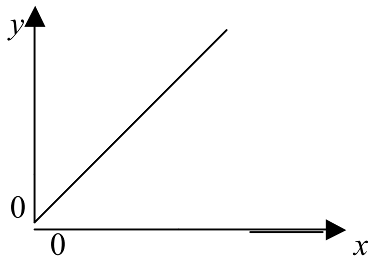


The region shown is the set of points (x,y) such that $0 < y < x < \infty$.

Consider the transformation of the boundary segments using:

$$u = x, v = \frac{y}{x}$$

13

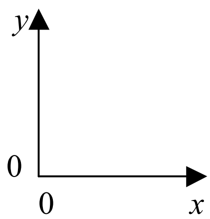


The region shown is the set of points (x,y) such that $0 < y < x < \infty$

Consider the transformation of the boundary segments using:

$$u = x, v = x - y$$

14



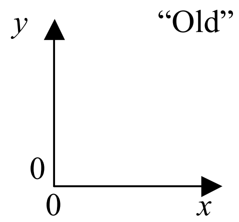
The quadrant shown is the set of points (x,y) such that $0 < x < \infty, 0 < y < \infty$.

Consider the transformation of the boundary segments using:

$$u = \frac{xy}{y+1}, v = \frac{x}{y+1}$$

In conclusion

Example 4



The quadrant shown is the set of points (x,y) such that $0 < x < \infty, 0 < y < \infty$.

Consider the transformation of the boundary segments using:

$$u = x + y, v = \frac{x}{x + y}$$

Starting at the origin and walking anticlockwise round the boundaries we have
- deferring consideration of (0,0) until return -

$\{y = 0, x \text{ increases from } 0 \text{ to } \infty\}$,

$\therefore \{u = x \text{ and } v = \frac{x}{x} = 1; u \text{ increases from } 0 \text{ to } \infty\}$

$\{x \to \infty, y \text{ increases from } 0 \text{ to } \infty\}$,

$\therefore \{u = x + y \to \infty, v = \dfrac{1}{1 + y/x} \to 1; u \to \infty,\ v = 1\}$

- an "old" side has shrunk to a "new" "point".

$\{x \to \infty, y \to \infty\}$

$\therefore \{u \to \infty, v \text{ is indeterminate but between } 0 \text{ and } 1\}$

- a "new" "side" is generated by an "old" "point".

$\{y \to \infty, x \text{ decreases from } \infty \text{ to } 0\}$,

$\therefore \{u = \to \infty, v = 0\}$,

- an "old" side has shrunk to a "new" "point".

$\{x = 0, y \text{ decreases from } \infty \text{ to } 0\}$

$\therefore \{u = y, v = 0 \ ; u \text{ decreases from } \infty \text{ to } 0 \text{ and } v = 0\}$

$\{x = 0, y = 0\}$

$\therefore \{u = 0, v \text{ is indeterminate but is between } 0 \text{ and } 1\}$

- a "new" "side" is generated by an "old" point.