# Liverpool University Maths Club <br> Circular Motion and the Cardioid Ian Porteous <br> Notes for 28 February, 2004 

## Warmups

You should begin the session by studying pages of the Funmaths Roadshow of the Liverpool Mathematical Society numbered
105. Chords Squared
122. Stepping Stones
140. A Circular Argument
191. Orthogonality

The first three of these involve circles, but in 105 and 122 one adopts the normal clock convention that one starts measuring angles from 'twelve o'clock', the clockwise direction being taken as positive, while 140 introduces the standard mathematical convention that one starts measuring angles from $(1,0)$, the positive direction being anticlockwise. In 122 one discovers that in doubling an angle it does not matter whether one goes clockwise or anticlockwise, and it is a nice first exercise to show why this is so.

## Squaring points

First we remark that by Pythagoras' theorem the circle in the $(x, y)$ plane with centre the origin and radius 1 has equation $x^{2}+y^{2}=1$. This circle is the unit circle.

We know how to square real numbers, represented by points of the real line. To square a point $(x, y)$ of the real coordinate plane the rule is to double the angle between the vector $(x, y)$ and the vector $(1,0)$ and at the same time square the distance to the origin. So by 140 the square of a point $(a, b)$ of the unit circle is $\left(a^{2}-b^{2}, 2 a b\right)$ and of the point $(r a, r b)$ is $r^{2}\left(a^{2}-b^{2}, 2 a b\right)=\left((r a)^{2}-(r b)^{2}, 2 r a r b\right)$, that is the square of any point $(x, y)$ of the plane is $\left(x^{2}-y^{2}, 2 x y\right)$.
Definitions of $\cos \theta$ and $\sin \theta$
The coordinates of the point of the unit circle at an angle $\theta$ are defined to be $(\cos \theta, \sin \theta)$. The square of $\cos \theta$ is written as $\cos ^{2} \theta$ and the square of $\sin \theta$ is written as $\sin ^{2} \theta$. It follows at once that, for any angle $\theta$,

$$
\cos ^{2} \theta+\sin ^{2} \theta=1, \quad \cos 2 \theta=\cos ^{2} \theta-\sin ^{2} \theta, \quad \sin 2 \theta=2 \cos \theta \sin \theta
$$

## Circular motion

Suppose that one traverses the unit circle at unit speed, measuring the distance gone round the circle from the angle 0 at $(1,0)$ to the angle $\theta$ at $(\cos \theta, \sin \theta)$ as the measure of the angle. This is known as the radian measure of the angle $\theta$, and we adopt this measure of angles from now on. The angle $\pi$ is half a full rotation, and the angle $2 \pi$ a full rotation.
Some values of the cosine and sine

$$
\begin{array}{cccccccccc}
\theta & 0 & \frac{1}{6} \pi & \frac{1}{4} \pi & \frac{1}{3} \pi & \frac{1}{2} \pi & \frac{2}{3} \pi & \frac{3}{4} \pi & \frac{5}{6} \pi & \pi \\
\cos \theta & 1 & \frac{1}{2} \sqrt{3} & \frac{1}{2} \sqrt{2} & \frac{1}{2} & 0 & -\frac{1}{2} & -\frac{1}{2} \sqrt{2} & -\frac{1}{2} \sqrt{3} & -1 \\
\sin \theta & 0 & \frac{1}{2} & \frac{1}{2} \sqrt{2} & \frac{1}{2} \sqrt{3} & 1 & \frac{1}{2} \sqrt{3} & \frac{1}{2} \sqrt{2} & \frac{1}{2} & 0
\end{array}
$$

## Derivative velocities

As one traverses the unit circle at unit speed the derivative velocity at the point $(\cos \theta, \sin \theta)$ is a unit vector orthogonal to the vector $(\cos \theta, \sin \theta)$, namely, by page 191 pf the Funmaths Roadshow, the vector $(-\sin \theta, \cos \theta)$, the speed at any point being the length of the velocity vector there.

It follows at once from this that the derivative velocity of $\cos \theta$ with respect to $\theta$ is $-\sin \theta$, while the derivative velocity of $\sin \theta$ with respect to $\theta$ is $\cos \theta$.

If one traverses the unit circle at twice unit speed the derivative velocity at the point $(\cos 2 \theta, \sin 2 \theta)$ will be $(-2 \sin 2 \theta, 2 \cos 2 \theta)$.


It would be good to have a dynamic demonstration of both motions at the same time.

## The cardioid

The circle $(x-1)^{2}+y^{2}=1$, with centre the point $(1,0)$ and radius 1 is given parametrically as

$$
(1+\cos \theta, \sin \theta)
$$

When this is described with unit velocity the derivative velocity at the point $\theta$ is the vector

$$
(-\sin \theta, \cos \theta)
$$

The curve obtained by 'squaring' this circle then is given parametrically by

$$
\begin{gathered}
\left(1+2 \cos \theta+\cos ^{2} \theta-\sin ^{2} \theta, 2 \sin \theta+2 \cos \theta \sin \theta\right) \\
=(1+2 \cos \theta+\cos 2 \theta, 2 \sin \theta+\sin 2 \theta)
\end{gathered}
$$

with derivative velocity

$$
(-2 \sin \theta-2 \sin 2 \theta, 2 \cos \theta+2 \cos 2 \theta)
$$

A computation shows that the speed at $\theta$ is $2 \sqrt{2} \sqrt{(1+\cos \theta)}$.

Try plotting the points of the curve and computing the derivative velocity and the consequent speed at these points for say

$$
\theta=0, \quad \frac{1}{3} \pi, \quad \frac{1}{2} \pi, \quad \frac{2}{3} \pi, \quad \pi
$$

Page 105 of the Funmaths Roadshow, turned on its side, is relevant here. The resultant heart-shaped curve is known as the cardioid. It has a cusp at the origin.


It would be nice to have an interactive computer demo here, showing at the same time the circle traversed at unit speed and how the speed along the cardioid then varies, reducing to zero at the cusp.

