MATHEMATICS CLUB HANDOUT SHEET

The Weird and Wonderful World of the Infinite

In Mathematics we are frequently confronted with the question what happens when things get infinitely large (or conversely when things get infinitesimally small). So common is this question that mathematicians have a special symbol

$$\infty$$

(something like a Mobius strip) to denote a number that is infinitely large. What I hope this session will demonstrate, is that when dealing with the infinite one most proceed with great care and learn not to trust what "common sense" would have us believe.

Section 1 - Infinite Series

Geometric Series

Consider the infinite series

$$1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\ldots\ldots$$

Take a little time to work out the following sums:

$$\begin{aligned}&1+1/2=\\&1+1/2+1/4=\\&1+1/2+1/4+1/8=\\&1+1/2+1/4+1/8+1/16=\end{aligned}$$

Can you guess what the value of the infinite sum might be?
Can you prove this? Spend a few minutes thinking about it.

HINT: We can write the sum of the first *n* terms of the series as

$$S_n=1+(1/2)+(1/2)^2+\ldots..+(1/2)^n$$

Try to find away of cancelling out most of the terms so that you have an expression for S_n which depends on $(1/2)^{n+1}$, and then consider what happens as *n* gets very large (or as *n* tends to infinity, $n\to\infty$, as mathematicians say)

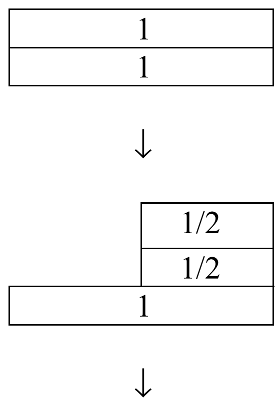
Using the same method as above, can you write down the sum of the first n terms of the general geometric series

$$S_n = 1 + r + r^2 + r^3 + \ldots\ldots + r^n$$

$$S_n =$$

What condition on r must apply to ensure that S_n tends to a fixed number as $n \to \infty$. Can r be negative?

Here is an interesting idea. Consider two rectangles of unit area as shown below. Divide the top one in two and place the two new rectangles, area ½, on top of each other.



Repeat this process a couple of times. How high is your tower that you are building on each case. Can you see the connection with our original geometric series? If you did this an infinite number of times how high would your tower be?

Explain this paradox when the your original area was only 2 square units?

Other types of series

From the above I hope you can appreciate that for the sum of an infinite series to **converge** to fixed value, the terms in the series must get smaller, i.e. they must tend to zero as $n \to \infty$. Consider the sum *S* of this interesting series (actually known as the harmonic series)

$$S = 1 + 1/2 + 1/3 + 1/4 + 1/5 + ...$$

The terms are obviously getting smaller, so would you like to hazard a guess at its eventual sum? Try working out

$$\begin{aligned}&1 + 1/2 = \\ &1 + 1/2 + 1/3 = \\ &1 + 1/2 + 1/3 + 1/4 = \\ &1 + 1/2 + 1/3 + 1/4 + 1/5 = \end{aligned}$$

Is there any pattern?

Perhaps in this case the sum *S* fails to converge to a fixed value. Can you prove this? Hint: The sum of the series

$$P = 1 + 1/2 + 1/2 + 1/2 + 1/2 + ... = \infty$$

i.e. it does not converge to a fixed limit but continues to grow indefinitely (because the terms are not getting smaller). Can you show that $S > P$? If so, then S must also continue to grow indefinitely.

What do you think happens if I change the harmonic series so that each even term becomes negative, i.e.

$$Q = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + ...$$

Does this series converge? Work out the sum of the first few terms. What sort of answers do you get?

What follows is a proof showing that $Q = 0$. Check it through carefully. I can rearrange the series by interchanging a few terms such that

$$Q = 1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \frac{1}{5} - \frac{1}{10} - \frac{1}{12} + \frac{1}{7} \ldots\ldots\ldots$$

I can gather together pairs of terms and rewrite this as

$$Q = \left(1 - \frac{1}{2}\right) - \frac{1}{4} + \left(\frac{1}{3} - \frac{1}{6}\right) - \frac{1}{8} + \left(\frac{1}{5} - \frac{1}{10}\right) - \frac{1}{12} + \left(\frac{1}{7} - \frac{1}{14}\right) \ldots$$

$$Q = \frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \frac{1}{10} - \frac{1}{12} + \frac{1}{14} \ldots$$

$$Q = \frac{1}{2}\left[1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \ldots\right]$$

$$Q = \frac{1}{2} Q$$

But the equation $Q = Q/2$ has only one solution, namely $Q = 0$. Proved.
Does this result agree with your calculations? If not, why not? Does my "proof" have a flaw?

INFINITE SETS

A set is a collection of objects that have some kind of inter relationship connecting them. For instance we might have the set *C* of African animals

$$C = \{\text{lion, crocodile, vulture}\}$$

and the set *P* of the class of animals they each belong to.

$$P = \{\text{bird, mammal, reptile}\}.$$

It is obvious each set has three members in this case. One way to see this is to pair off one member of each set with each other like this.

$$\begin{aligned}&\text{lion} \longleftrightarrow \text{mammal}\\ &\text{crocodile} \longleftrightarrow \text{reptile}\\ &\text{vulture} \longleftrightarrow \text{bird}\end{aligned}$$

We can do the same things with sets of numbers. Consider the two sets of integers $A = \{1, 2, 3, 4, 5, 6\}$ and $B = \{5, 7, 9, 11, 13, 15\}$. It is obvious each has six elements. Can you pair off the numbers from set *A* with a single member from set *B*. What is the rule connecting them?

$$\begin{aligned}&1 \longleftrightarrow\\ &2 \longleftrightarrow\\ &3 \longleftrightarrow\\ &4 \longleftrightarrow\\ &5 \longleftrightarrow\\ &6 \longleftrightarrow\\ &n \longleftrightarrow\end{aligned}$$

We say that there is a one-one correspondence between the members of each set. By invoking the rule we can take any element of set *A* and move (map) to a **unique** element of set *B*. By using the **inverse** rule we can map back from an element of set *B* to an element of set *A*.

Now let us consider two infinite sets, *N* the set of all natural numbers and 2*N* the set of positive even numbers.

$$\begin{aligned}N &= \{1, 2, 3, 4, 5, 6, ...\}\\ 2N &= \{2, 4, 6, 8, 10, 12, ...\}\end{aligned}$$

Can we ask the question do the two sets have the same number of elements?

Of course the answer is no, isn't it? After all common sense says the set *N* contains all the odd integers 1, 3, 5, 7,…. which are missing from the set 2*N*, so it must have more

elements. Except that *N* has an infinite number of elements, so that we might guess $2N$ has $\infty/2$ elements, but $\infty/2$ is still infinity!
How can we get around this? Well if we return to our idea of pairing off elements in each set, we can see that every element *N* is matched with its double:

$$
\begin{array}{c}
1 \longleftrightarrow 2 \\
2 \longleftrightarrow 4 \\
3 \longleftrightarrow 6 \\
4 \longleftrightarrow 8 \\
5 \longleftrightarrow 10 \\
\vdots \\
n \longleftrightarrow 2n
\end{array}
$$

From this matching we can see that for every element of *N* there is a corresponding element of 2*N*, all the naturals match up with *all* the evens and hence the two sets have the *same* number of elements (the sets have the same *cardinality*). Note the correspondence *only* works if we are dealing with the sets as *completed wholes*. If we limit the set *N* to the first 100 natural numbers, the 1-1 correspondence breaks down because there only 50 evens less than or equal to 100, and 50 odds. I warned you "common sense" must be suspended when dealing with ∞!

Can you find a 1-1 correspondence between the set of natural numbers *N* and the following infinite sets, and so prove they have the same number of elements (cardinality)?

Set of odds $O = \{1, 3, 5, 7, 9, ...\}$
Set of Squares $S = \{1, 4, 9, 16, 25, 36, ...\}$
Set of all integers $Z = \{....., -4, -3, -2, -1, 0, 1, 2, 3, 4, ...\}$

$$
\begin{array}{cccccc}
N & & & O & S & Z \\
1 & \longleftrightarrow & & & & \\
2 & \longleftrightarrow & & & & \\
3 & \longleftrightarrow & & & & \\
4 & \longleftrightarrow & & & & \\
5 & \longleftrightarrow & & & & \\
6 & \longleftrightarrow & & & & \\
\vdots & \vdots & & & & \\
n & \longleftrightarrow & & & &
\end{array}
$$

OK you may say I can just about believe that there are the same number of integers *Z* as natural numbers *N*. But can we take this idea further? Take the set of rational numbers *Q*, that is all the positive whole numbers and the fractions. Surely *Q* must

contain more elements than N? After all just between the numbers 1 and 2 there are infinitely many fractions, for instance the subset $\{3/2, 5/4, 7/6, 9/7, 11/9, ...\}$ all lie in this range.
HOWEVER, when dealing with ∞, "common sense" we know just does not work.
Let us see if we can we create a 1-1 correspondence and the natural numbers and the rationals. Let us list the latter systematically

$$\frac{1}{1} \quad \frac{2}{1} \quad \frac{3}{1} \quad \frac{4}{1} \quad \frac{5}{1} \quad \frac{6}{1} \quad \frac{7}{1} \quad \frac{8}{1} \quad \frac{9}{1} \cdots$$

$$\frac{1}{2} \quad \frac{2}{2} \quad \frac{3}{2} \quad \frac{4}{2} \quad \frac{5}{2} \quad \frac{6}{2} \quad \frac{7}{2} \quad \frac{8}{2} \quad \frac{9}{2} \cdots$$

$$\frac{1}{3} \quad \frac{2}{3} \quad \frac{3}{3} \quad \frac{4}{3} \quad \frac{5}{3} \quad \frac{6}{3} \quad \frac{7}{3} \quad \frac{8}{3} \quad \frac{9}{3} \cdots$$

$$\frac{1}{4} \quad \frac{2}{4} \quad \frac{3}{4} \quad \frac{4}{4} \quad \frac{5}{4} \quad \frac{6}{4} \quad \frac{7}{4} \quad \frac{8}{4} \quad \frac{9}{4} \cdots$$

$$\vdots$$

Can you devise a method to pair off all the fractions with all the natural numbers, in a 1-1 correspondence? If so we will have proved that N and Q have the same number of elements.

$$\begin{array}{ccc} N & & Q \\ 1 & \longleftrightarrow & \\ 2 & \longleftrightarrow & \\ 3 & \longleftrightarrow & \\ 4 & \longleftrightarrow & \\ 5 & \longleftrightarrow & \\ 6 & \longleftrightarrow & \\ 7 & \longleftrightarrow & \\ 8 & \longleftrightarrow & \\ & \vdots & \end{array}$$

Hint: Think diagonally!

Exercise: Can you extend this idea to include the negative fractions as well?

Up to now we have shown there exists a 1-1 correspondence between the natural numbers N, the set of all integers Z and the rational numbers (integers and fractions) Q. Each set has an infinite number of elements, but because we can pair off each member of one set with a member from another, we know that the 'infinite number of elements' actually represents the same quantity.

OK so what is left. Well obviously what happens if we look at the set of all real numbers, that is the set of integers, rationals (fractions) and irrationals. An irrational number is one that cannot be written as a ratio of two natural numbers. One such example is $\sqrt{2} = 1.4142135\ldots$. Perhaps you can think of some others?

Why is $\sqrt{2}$ a different from type of number from the integers and fractions we are so familiar with. For instance, can we prove $\sqrt{2}$ is not rational? Well suppose on the contrary let us assume that it is. This would mean that we could write $\sqrt{2}$ in the form

$$\sqrt{2} = \frac{a}{b}$$

where *a* and *b* are two natural numbers *with no common factors*. Can you prove this is impossible? Hint: Square the equation and show *a* and *b* are even, contradicting the fact they have no common factors.

OK so $\sqrt{2}$ is definitely different from other numbers like integers and fractions. However, although the set of all reals $R = \{1, 1/2, \sqrt{2}, \ldots\ldots\ldots\ldots\}$ is infinite, that does not mean in reality it is any 'bigger' than the sets *N*, *Z* and Q, which we have met earlier. In fact all we have to do to show that *R* is the same 'size' as *N* is to pair off the elements of each to form 1-1 correspondence. The thing about the reals is that they can all be written as infinite decimals, which either terminate into nothing but zeros (e.g. 1=1.000000…., ½=0.5000000…) or repeat a sequence (1/3=0.3333333…., 1/7=0.142857142857142857….) or, if they are irrational like $\sqrt{2}$=1.41421356…, result in a decimal form with a completely random pattern. Can you write down the decimal expansions for

$$1/13 =$$
$$1/9 =$$
$$1/5 =$$
$$2/3 =$$
$$1/\sqrt{2} =$$

Which is irrational and why?

Notice all these numbers lie between 0 and 1. Let us construct another real number also between 0 and 1, which is distinct from these five. We shall do so using the following recipe.

1. Start by writing 0 point.
2. Look at the first number in our list. Look at the first decimal place. If the number in that position is 5 write 6, otherwise write 5.
3. Look at the second number in the list. Look at the second decimal place. Again if the number in that position is a 5 write 6, otherwise write 5.
4. Repeat going down all the numbers in the list
5. Add an infinite set of zeros at the end.

What number do you get? Is it different from the other five?

Why is this relevant? Well we know that to prove *R* has the same number of elements as *N* (i.e. these two infinite sets are the same 'size') we must find a 1-1 correspondence between them. Let us restrict our search to real numbers between 0 and 1. A 1-1 correspondence would look like

$$\begin{array}{l}
1 \leftrightarrow 0.a_{11}a_{12}a_{13}a_{14}a_{15}a_{16}\ldots\ldots \\
2 \leftrightarrow 0.a_{21}a_{22}a_{23}a_{24}a_{25}a_{26}\ldots\ldots \\
3 \leftrightarrow 0.a_{31}a_{32}a_{33}a_{34}a_{35}a_{36}\ldots\ldots \\
4 \leftrightarrow 0.a_{41}a_{42}a_{43}a_{44}a_{45}a_{46}\ldots\ldots \\
5 \leftrightarrow 0.a_{51}a_{52}a_{53}a_{54}a_{55}a_{56}\ldots\ldots \\
6 \leftrightarrow 0.a_{61}a_{62}a_{63}a_{64}a_{65}a_{66}\ldots\ldots \\
\vdots \qquad\qquad \vdots \\
\vdots \qquad\qquad \vdots \\
\vdots \qquad\qquad \vdots
\end{array}$$

Here a_{mn} represents the digit in the *n*th decimal place of the *m*th real between 0 and 1. Obviously our list is infinite, but we know on the right hand side contains *all* the reals between 0 and 1 written in decimal form.

However, what happens if we apply the recipe described above to this list? What sort of number do we create? Is it in our list??

What does this say about the number of reals *R* compared to the number of natural numbers *N*? Can we have 'infinities' of different sizes??

Further reading: Much of this work is discussed in "The Art of the Infinite", by Robert and Ellen Kaplan, published by Allen Lane (Penguin Books) 2003. There you can find many more strange ideas about the infinite.