LATIN SQUARES AND GROUPS
Maths Club

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LATIN SQUARES

A latin square is an array of elements in which each element appear once and only once in every column and row. For example if we have four objects lets call them

$$A, B, C, D,$$

then one possibility for a latin square would be:

A	B	C	D
B	C	D	A
C	D	A	B
D	A	B	C

Now if we take the same four objects can you arrange them to create two new latin sqaures different to the one above?

Example 1

A	B	C	D

A	B	C	D

Example 2

Now we are going to use the idea of a latin square to fill in some multiplication tables. In this example we are going to complete the multiplication table for the following four elements

$$\{1, A, B, C\}.$$

Such that the following relations hold

$$1.A = A = A.1, \quad 1.B = B = B.1, \quad 1.C = C = C.1,$$

and

$$A^2 = A.A = B,$$
$$A.B = C = B.A,$$
$$A.C = 1 = C.A.$$

	1	A	B	C
1				
A			**C**	
B				
C				

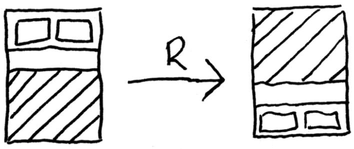
Note: When multiplying we take the elements in a certain order, the first element comes from the left-hand column while the second is from the top row. For example if we take the multiplication $A.B$ we would take the row corresponding to A then we would take the column related to B. The multiplication $A.B$ from the relations gives C which has already been placed in the table for you.

Example 3

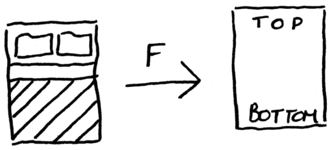
This time we are going to think of the elements as translations or movement of an object.

So I would like you to think about a mattress, for a mattress to stay in good condition It needs to be flipped or rotated (so that it gets equal wear). There are three possible movements:

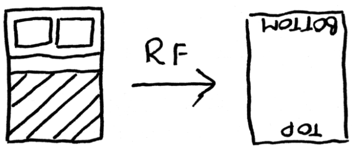
1. Mattress can be rotated 180° anticlockwise. We shall call this R.



2. Mattress can be flipped over so that the top and bottom stay the same. We shall call this F.



3. Or we can flip from top to bottom, but this is the same as a rotation (R) followed by a flip (F); we shall call this $RF = FR$



Now with this information fill in the following table:

	1	R	F	RF
1				
R				
F				
RF				

Example 4

This time the elements of the set are four matrices, and the operation is matrix multiplication.

$$1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}; \quad X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix};$$

$$Y = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}; \quad Z = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix};$$

So we must multiply the matrices to fill the table:

	1	X	Y	Z
1				
X				
Y				
Z				

Examples

$$1.X = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} & \\ & \end{pmatrix}$$

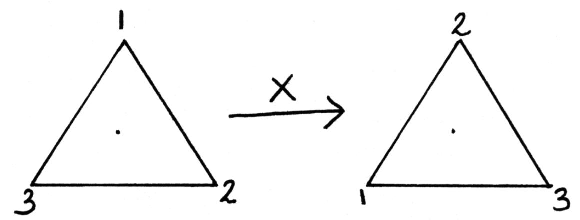
$$X.X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} & \\ & \end{pmatrix}$$

$$Y.Z = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} & \\ & \end{pmatrix}$$

Example 5

Symmetries of a Triangle

Let us first start with the rotations or an equilateral triangle:

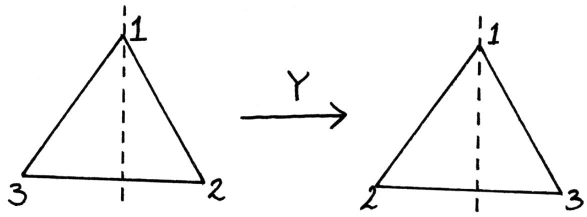


1. Let X be ONE rotation of $120°$ about the centre anti-clockwise.
2. Let $X.X = X^2$ be TWO rotations of $120°$ about the centre anti-clockwise.
3. Let $X.X.X = X^3$ be THREE rotations of $120°$ about the centre anti-clockwise which will take us back to our original position.

	1	X	X^2
1			
X			
X^2			

As well as rotational symmetry we also have reflectional symmetry, which we wil call Y. In the case of our equilateral triangle the lines of reflection are the lines joining the midpoint of any side to the opposite vertex.

So for example we have:



Now using this information we know that there will be six different ways to rotate or reflect an equilateral triangle, and thus we can create our operation table as before. *Note: Unlike the other tables here we don't have symmetry down the main diagonal this is because $XY \neq YX$.*

	1	X	X^2	Y	XY	X^2Y
1						
X						
X^2						
Y						
XY						
X^2Y						

GROUPS

So what is a **GROUP**? Well belive it or not you have been working with groups for the past half hour, we can define a group as follows:

Definition

A group is a set of numbers, letters, etc., together with an operation $\circ$ satisfying the four axioms:

(G1) The set is **closed** under the operation $\circ$.

(G2) The operation $\circ$ is **associative**.

(G3) An **identity element** exists and is unique.

(G4) Each element of the set has a unique **inverse**.

As we are all mathematicians we like to put simple definitions into strange notation. So we can aslo define a group in this more abstract way.

Definition

A group is a set G together with an operation $\circ$ satisfying the four axioms:

(G1) $x \circ y \in G$ for $x, y \in G$,

(G2) $x \circ (y \circ z) = (x \circ y) \circ z$ for $x, y, z \in G$,

(G3) there exists an element $e \in G$ such that for all $g \in G$ $e \circ g = g = g \circ e$,

(G4) for every element $g \in G$ there exists an element $g^{-1} \in G$ such that $g \circ g^{-1} = e = g^{-1} \circ g$.

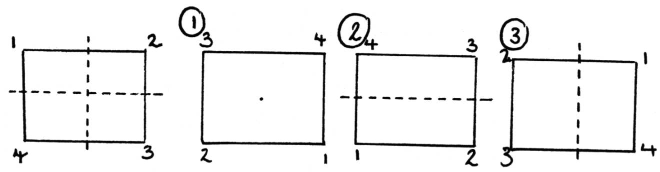
Don't worry though, all this means is that if we have an operations table that is a latin square then it is likely to be a group.

DIHEDRAL GROUP

Given a geometric object (in the plane) we are going to consider transformations of the plane which fix our object. If we consider a non-square rectangle.

1. then we have rotation of 180° about the origin (anti-clockwise),
2. we have reflection about the horizontal axis, also
3. we have reflection about the vertical axis (this is the same as a one rotation followed by a horizontal reflection).

DIAGRAM



We can create a multiplication table for these symmetries.

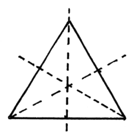
	1	A	B	C
1	1	A	B	C
A	A	1	C	B
B	B	C	1	A
C	C	B	A	1

These symmetries form a group (under the operation of composition of symmetries), the above operations table proves that they form a group as all four group axioms (rules) hold. In fact you have meet this group before, this is the same group as the flipping of the mattress!

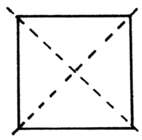
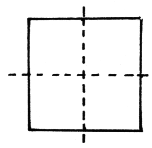
We can extend this idea to a regular n-sided polygon. We have obvious symmetries obtained by rotating our polygon (anticlockwise) through an angle of $360°/n$. If we let x denote this basic rotation then $x^n = 1$ (the identity rotation).

There are also n reflections:

1. When n is odd the lines of reflections are the lines joining the midpoint of any side to the opposite vertex.



2. When n is even the lines of reflection are the lines that join the midpoint of any side to the midpoint of its opposite side and those that join vertex to opposite vertex.

These symmetries form a group called the **dihedral group**, denoted D_n with $2n$ elements.

EXAMPLE We have already seen two examples of the dihedral group, the flipping of the mattress and the symmetries of an equilateral triangle. Now let us consider the symmetries of a square.

1. Let X denote a rotation of $90°$ anti-clockwise.
2. Let Y denote a reflection.

Fill in the following table such that the following dihedral relations hold

$$X^4 = 1, \quad Y^2 = 1, \quad (X^iY)^2 = 1,$$

and

$$X^iY = YX^{-i}, \quad (\text{e.g. } XY = YX^{-1} = YX^3).$$

	1	X	X^2	X^3	Y	XY	X^2Y	X^3Y
1								
X								
X^2								
X^3								
Y								
XY								
X^2Y								
X^3Y								

PERMUTATION GROUPS

The permutations (or rearrangement) of n different elements form a group. We write elements of the group in this two row notation. The top row denotes the starting position, while the bottom row denotes where the elements end up.

$$\begin{pmatrix} 1 & 2 & \dots & n \\ \downarrow & \downarrow & & \downarrow \\ x_1 & x_2 & \dots & x_n \end{pmatrix}$$

For example the following permutation means, 1 goes to 2, 2 goes to 3 and 3 goes to 1.

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \quad \begin{array}{ccc} 1 & \rightarrow & 2 \\ 2 & \rightarrow & 3 \\ 3 & \rightarrow & 1 \end{array} \quad (1\ 2\ 3)$$

We can shorten the two row notation to a simpler one row notation so (1 2 3) means the same. Let us consider all the permutations of three elements (there are $3! = 6$ possible permutations).

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}.$$

These six permutations for a group under an operation, $\circ$, which we will show be means of an example. Compose these two permutations together:

$$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} = (13), \quad \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} = (12).$$

$$(13) \circ (12) = \begin{pmatrix} 1 & 2 & 3 \\ \downarrow & \downarrow & \downarrow \\ 3 & 2 & 1 \\ \downarrow & \downarrow & \downarrow \\ 2 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = (123)$$

To show that these six permutations form a group we can produce our operations table.

	1	(123)	(132)	(12)	(23)	(13)
1	1	(123)	(132)	(12)	(23)	(13)
(123)	(123)	(132)	1	(13)	(12)	(23)
(132)	(132)	1	(123)	(23)	(13)	(12)
(12)	(12)	(23)	(13)	1	(123)	(132)
(23)	(23)	(13)	(12)	**(132)**	1	(123)
(13)	(13)	(12)	(13)	(123)	(132)	1