Magic Squares of order 4

It has been known since 1693 that there are (up to rotations and reflections) 880 such squares, Frénicle having published a list in that year. The sources that have assisted me to verify the list were:

- ‘Amusements in Mathematics’ by H. E. Dudeney, pages 119 et seq
- ‘Mathematical Recreations’ by M. Kraitchik, pages 183 et seq and pages 322-2

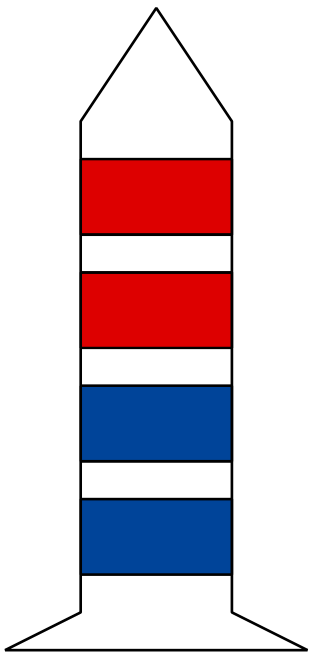
We start by looking at that last reference, which does not explicitly mention magic squares at all! Our other source materials are Dudeney’s list of the 12 types into which the squares may be classified, and a list, derived from a list of Kraitchik, of all the magic series of four integers from 0 to 15, summing to 30.

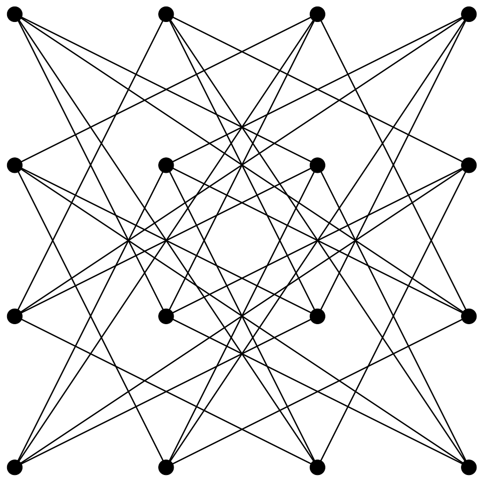
Ian <porteous@liv.ac.uk>

The Four-Storey Towers

G. Kowalewsky gives the following description of his game: The pieces are little four-storey towers, each storey represented by a horizontal band painted either red or blue. Every possible arrangement of red and blue bands is represented, so there are $2^4 = 16$ different towers. The game is played on a 4×4 chessboard.

Two towers are said to be “in domino” if they differ in the colour of just one storey. The object of the game is to place all 16 towers on the board, one after the other, so that every pair that is separated by a knight’s move (1, 2) or by a move (2, 3) shall be in domino (see diagram below).

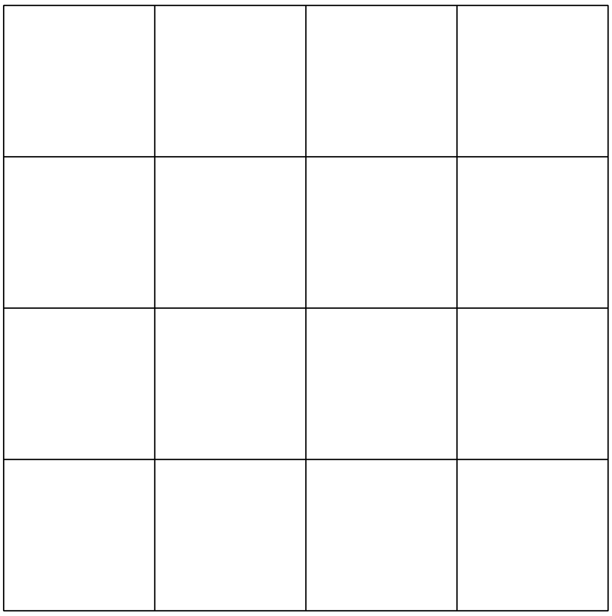




The 16 towers may be represented by the first 16 binary integers:

0000	0001	0010	0011
0100	0101	0110	0111
1000	1001	1010	1011
1100	1101	1110	1111

Cut them out, and play on the board below.



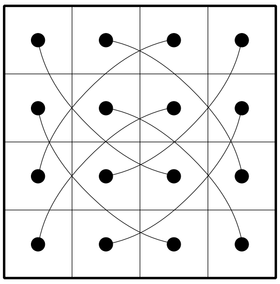
For reasons that should be apparent we form squares using the numbers 0 to 15, rather than the more traditional 1 to 16. In such a square each row, each column and each diagonal has 30 as its sum.

From any square we may derive another by interchanging, or ‘flipping’, two adjacent columns, and then interchanging, or ‘flopping’, the corresponding two rows. We refer to this as performing a ‘flip-flop’. As we shall later verify, the result will still be a magic square if the columns and rows interchanged are the central ones.

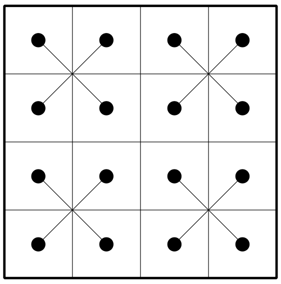
If we divide any square into quarters and then interchange the top left and bottom right quarters, and then the top right and bottom left quarters, we obtain another square. We call this operation ‘quartering’ the square. As we shall later verify the result of quartering will still be a magic square.

The following diagram shows the twelve types of square of order 4, classified by the patterns made by conjugate pairs of numbers, that is numbers adding up to 15, half the magic total of 30.

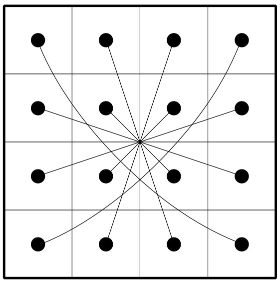
Type I



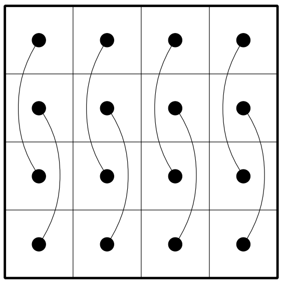
Type II



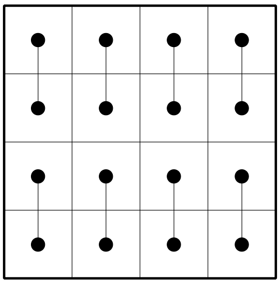
Type III



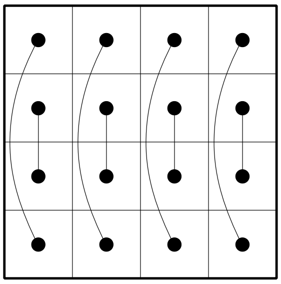
Type IV



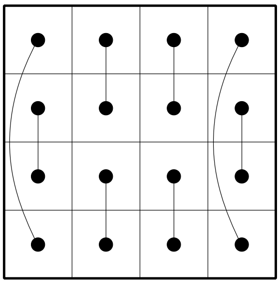
Type V



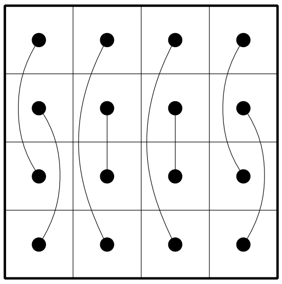
Type VI



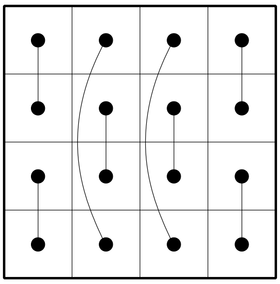
Type VII



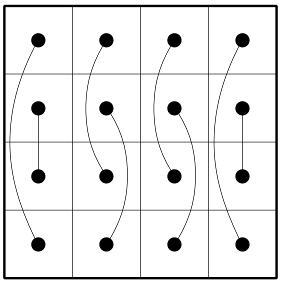
Type VIII



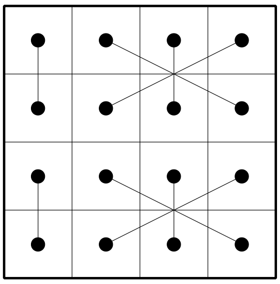
Type IX



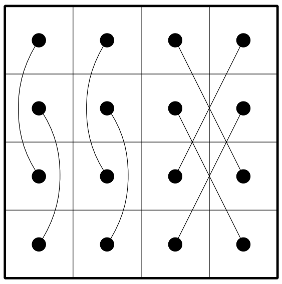
Type X



Type XI



Type XII



Programme

- Verify that any square of order 4 can be converted to another by (central) flip-flop and/or quartering.
- Using ‘four-storey towers’, that is the whole numbers from 0 to 15 in binary form, construct a square of type I (a ‘Nasik’ square), and verify that it is ‘regular’, that is its semi-diagonals also are magic, and verify moreover that *any* split diagonals of such a square are magic. Determine how many such squares there are.
- Verify that squares of types II and III may be obtained from squares of type I by flip-flop.
- Verify that squares of types IV, V and VI may be obtained by ‘reaming’ (replacing either rows or columns by diagonals and semi-diagonals) squares of types I, II and III.

Example:	0	7	12	11	goes to	0	10	15	5
	13	10	1	6		13	7	2	8
	3	4	15	8		3	9	12	6
	14	9	2	5		14	4	1	11

- Determine how many regular squares you now have.
- Show how some more non-regular squares of type VI may be derived from regular squares of type VI by ‘tweaking’, that is by swapping the two middle numbers in each of the two outside columns, or in each of the two outside rows.

This is the half-way stage. What remains to be done is maybe even more interesting!

Magic Series

The following lists give all the order 4 magic series in decimal form, but the classification refers to them in binary form.

A series is ‘regular’ if when its terms are in binary form then, given any number in the series, two of the others differ from it in all digits except one, while the other agrees with it in exactly two digits, there being two 0’s and two 1’s at each level.

A series is ‘symmetric’ if it consists of two conjugate pairs, being ‘regular symmetric’ if any number of either pair agrees with any number of the other pair in exactly two digits.
A series is ‘irregular’ otherwise.

Regular series [0 2 3 3] :

0	3	13	14	0	5	11	14
0	6	11	13	0	7	9	14
0	7	10	13	0	7	11	12
1	2	12	15	1	4	10	15
1	6	8	15	1	6	10	13
1	6	11	12	1	7	10	12
2	4	9	15	2	5	8	15
2	5	9	14	2	5	11	12
2	7	9	12	3	4	8	15
3	4	9	14	3	4	10	13
3	5	8	14	3	6	8	13
4	7	9	10	5	6	8	11

Regular symmetric series [0 2 2 4] :

0	3	12	15	0	5	10	15
0	6	9	15	1	2	13	14
1	4	11	14	1	7	8	14
2	4	11	13	2	7	8	13
3	5	10	12	3	6	9	12
4	7	8	11	5	6	9	10

Semi-regular symmetric [0 1 3 4] :

0	1	14	15	0	2	13	15
0	4	11	15	0	7	8	15
1	3	12	14	1	5	10	14
1	6	9	14	2	3	12	13
2	5	10	13	2	6	9	13
3	4	11	12	3	7	8	12
4	5	10	11	4	6	9	11
5	7	8	10	6	7	8	9

Irregular series :

*	0	4	12	14	0	5	12	13
*	0	6	10	14	0	8	9	13
*	0	8	10	12	0	9	10	11
	1	3	11	15	1	4	12	13
	1	5	9	15	1	5	11	13
	1	7	9	13	1	8	9	12
	1	8	10	11	2	3	10	15
	2	3	11	14	2	4	10	14
	2	6	7	15	2	6	8	14
	2	6	10	12	2	7	10	11
	2	8	9	11	3	5	7	15
	3	5	9	13	3	6	7	14
	3	6	10	11	3	7	9	11
	3	8	9	10	4	5	6	15
	4	5	7	14	4	5	8	13
	4	5	9	12	4	6	7	13
	4	6	8	12	5	6	7	12

The list of all magic series gives them in decimal form. In each of the regular series, or semi-regular series, when in *binary form*, you will find that on each layer of the 'towers' there are exactly two 0's and two 1's. For an irregular series this is no longer the case.

For a regular series any permutation of the layers produces another regular magic series. For irregular series, on the other hand, an arbitrary permutation of the layers will produce a new series that is not magic. Yet certain permutations may preserve the magic property, and this may be used as a means of classifying the irregular series. Series that can be derived the one from the other by a permutation of the binary layers will be said to be of the same 'layer type'. It turns out to be the case that for any magic square of order 4 the two diagonals of the square are of the same layer type.

- Three of the irregular series have been starred. Determine all the magic series layer-equivalent to each of the starred series, and then proceed to sort all the irregular series into three layer types.
- Verify that squares of types VII to X may be derived from each other by (central) flip-flop and/or by quartering. Likewise verify that squares of types XI and XII may be derived from each other by (central) flip-flop.
- For each of the starred irregular series find for which permutations of the series, if any, there is a square of type VI, VII or XI, with that series as one of its diagonals. Also in each case determine whether a further square can be derived from the given one by 'tweaking'.
- Complete the classification of all 880 magic squares of order 4, stating the total number of squares of each type.
-

Information Concerning the Starred (*) Irregular Series:

0	**4**	**12**	**14**	transforms to	0	8	9	13
0	0	0	0		0	0	1	1
0	0	0	1		0	0	0	0
0	1	1	1		0	0	0	1
0	0	1	1		0	1	1	1

the 'bottom row' in the binary representation moving up to the top.

There are 4 such pairs. They form the diagonals of squares of all the types VI – XII.

0	**6**	**10**	**14**	transforms to	0	12	5	13	and to	0	9	10	11
0	0	0	0		0	0	1	1		0	1	0	1
0	1	1	1		0	0	0	0		0	0	1	1
0	1	0	1		0	1	1	1		0	0	0	0
0	0	1	1		0	1	0	1		0	1	1	1

the ‘bottom row’ moving up to the top, then the same happening again.

There are 8 such triples. They form the diagonals of squares of types VII to X only.

0	**8**	**10**	**12**	is a singleton, as is its conjugate	3	5	7	15
0	0	0	0		1	1	1	1
0	0	1	0		1	0	1	1
0	0	0	1		0	1	1	1
0	1	1	1		0	0	0	1

There are just these two. They form the diagonals of squares of types VI only.

This accounts for all 34 = 8 + 24 + 2 irregular magic series.

In almost all cases the order in which each series is placed in the diagonal matters. It turns out that in every case both diagonals have to be of the same type, though it is only for squares of type VI that either diagonal is the conjugate of the other.

Each row and each column of any magic square of order 4 is either a regular magic series, or a symmetric magic series. The only series that never occur as diagonals are the semi-regular symmetric series.

The Tally

There are 16 × 24 ways of placing the four-storey towers, and in each case we finish up with a magic square of type I, but the number of distinct such squares is this number divided by 8, that is 48. These give rise to 48 squares of types II and III and of twice that number, namely 96, squares of types IV, V and VI. Thus we have 432 regular squares in all. By tweaking each square of type VI we get a further 96 semi-regular squares.

There remain 880 - 432 - 96 = 352 irregular squares. These arise as follows.

We have seen that the 34 irregular magic series fall into 8 triplets, 4 pairs and two singletons. Each gives rise to magic squares with that series in some order or other as one of its diagonals, but the number of possible orders in which they may lie along the diagonal varies with each type. Each of the 24 series that form *triplets* gives rise to four distinct squares of each of the types VII to X with that series as one of its diagonals, but no squares of any of the other types, but each square has two diagonals, so we have to divide the total by two. That gives us 192 squares in all. Each of the 8 series that form *pairs* gives rise to 16 distinct squares of type VI as one of its diagonals, but only 2 of each of the types VII to X, and just 2 of each of the types XI and XII. Again dividing the total by two that gives 64 + 8 + 8 + 8 + 8 + 8 + 8 = 112 in all. Finally, and rather surprisingly, either of the singletons may be entered on a diagonal in any order whatsoever, and then be completed to a square of type VI in four ways, the other diagonal being the conjugate magic series. Opposite orders lead to equivalent squares. This provides 48 distinct squares in all. So finally we have 192 + 112 + 48 = 352 in total, as previously asserted.

It is another matter altogether to prove that there are no other squares. I do not know where this was first completely established.

Table of results

I		48		
II		48		
III		48		
IV		96		
V		96		
VI		96		432
VI				96
	Triplets	Pairs	Singletons	
VI	0	64	48	112
VII	48	8	0	56
VIII	48	8	0	56
IX	48	8	0	56
X	48	8	0	56
XI	0	8	0	8
XII	0	8	0	8
				880