MATHEMATICS CLUB HANDOUT SHEET

The Modulus Function $y = |x|$, and Some Applications

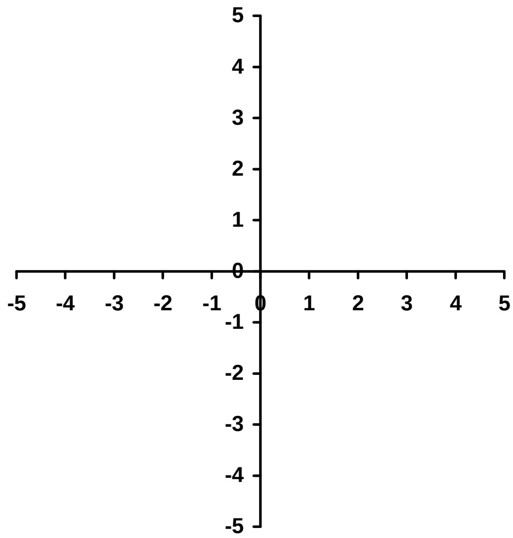
Let us consider the function

$$y = |x|,$$

where $|x|$ means the **absolute value** or **modulus** of the number *x*. That is

$$|x| = \begin{cases} x, & \text{if } x > 0 \\ 0, & \text{if } x = 0 \\ -x, & \text{if } x < 0 \end{cases} .$$

Exercise 1. Can you draw the graph of $y = |x|$? On the scales below plot the graph of $y = |x|$, for x lying between -5 and $+5$.

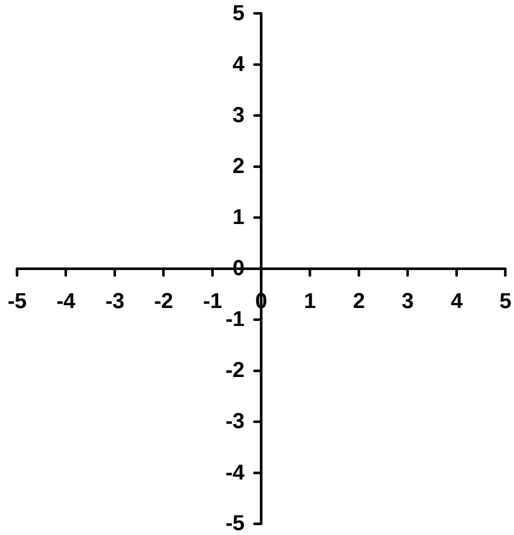


What sort of graph do you get? Is it a line or a curve?
Is *y* ever negative?
What angle does the line make with the *x*-axis?
How is the graph for positive *x* values related to the graph for negative *x* values?

Graphs like this which are symmetric in the *y*-axis are known as **even** functions of *x*.
Even functions are defined by the property

$$y = f(x) = f(-x),$$

where here *f* stands for any function of *x*. For instance $y = x^2$ is an even function of *x* as $y = x^2 = (-x)^2$. Plot the graph of $y = x^2$ on the axes below. Is it symmetric in the *y*-axis?

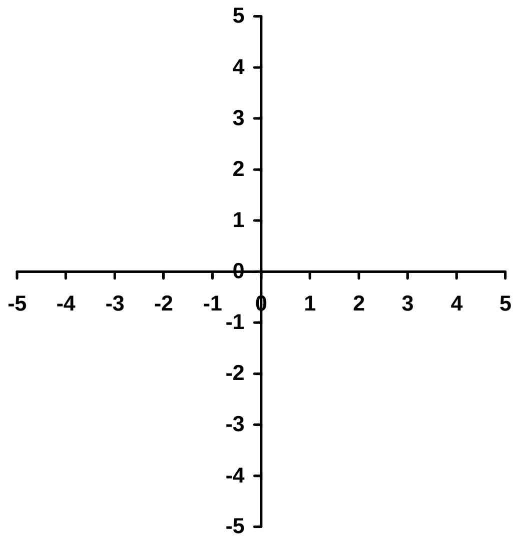


Exercise 2 Can you plot the graphs of $y = |x| + 1$ and $y = |x| - 1$. To help you, fill in the table of *y* values below.

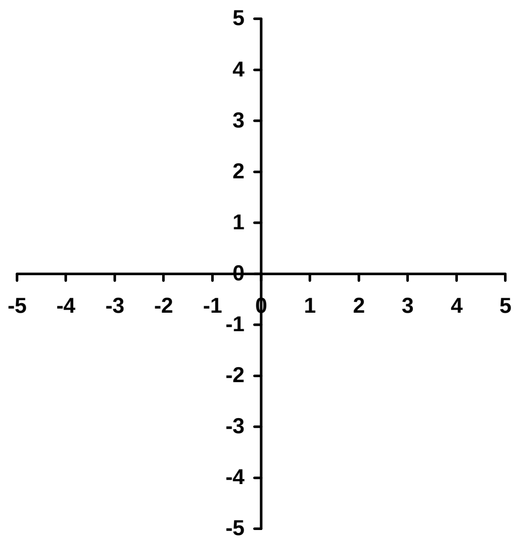
x	$y = \lvert x \rvert + 1$	$y = \lvert x \rvert - 1$	$y = \lvert x + 1 \rvert$
-5			
-4			
-3			
-2			
-1			
0			
1			
2			
3			
4			
5			

Now can you plot the graphs? Plot them both on the axes overleaf.
What do you have to do to your original graph of $y = |x|$**,** to obtain these two new graphs?

The procedure you have just described is called a **translation** (which is just a special word for a movement) along the *y*-axis. Many graphs can be obtained by using translations. For instance we can plot the curve of the graph $y = 2 + x^2$, from our knowledge of the curve $y = x^2$, simply by translating the curve of the latter 2 units up along the *y*-axis.



Exercise 3. Can you plot the graph of $y = |x+1|$? To help, first calculate the *y* values for $x = -5, -4, \ldots\ldots, 4, 5$ and list them in the table on the previous page. Then using these values plot the graph on the axes below.



How is this graph obtained from the original graph for $y = |x|$?
What sort of translation is involved?
Can you, without calculating any values, draw the graph of $y = |x-1|$ on the axes above?
What sort of translation is required this time?

Exercise 4 More applications of the modulus function. On the attached sheet the graph of the curve $y = x^4 - 5x^2 + 4$ is plotted. Is this an even function?

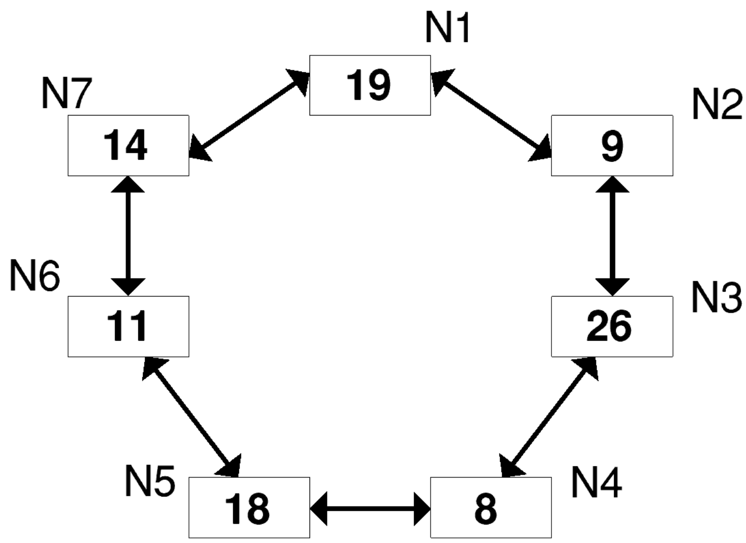
On the same graph can you plot $y = \left|x^4 - 5x^2 + 4\right|$, *without* doing any calculations?

On the second attached sheet there is a plot of the curve $y = \dfrac{x}{x^2+1}$. On the same graph can you draw the following curves?

$$y = \left|\frac{x}{x^2+1}\right|, \qquad\qquad y = \frac{|x|}{x^2+1}.$$

Harder Exercises for the Keen Mathematicians

1. Find the minimum value of the function $y = |x-2| + |x| + |x+2| + |x+4|$. *Hints:* Consider carefully what happens at the points $x = -4, -2, 0,$ and 2. Plot the function over the ranges $x < -4$, $-4 \le x < -2$, $-2 \le x < 0$, $0 \le x < 2$ and $x \ge 2$. (A table of values is useful.) From your graph you should be able to determine the minimum.
2. Seven matchboxes are arranged in a circle as shown. The first contains 19 matches, the second 9 matches, and the remaining ones contain 26, 8, 18, 11 and 14 matches, respectively. Matches may be taken from any box and put into an adjacent box. The matches must be shifted in such a way that the number of matches in all boxes becomes the same. How can this be done, shifting as few matches as possible?



Hints: First determine how many matches each box must contain for them to all have the same amount. Now suppose the number of matches that are moved from box 1 to box 2 (say) to give this number, is equal to *x*. (Note *x* could be negative, in which case we must shift the matches the other way, from box 2 to box 1.) Now box 2 contains $9 + x$ matches. How many matches must be moved from box 2 to box 3 to give the number we are interested in? How many matches are now in box 3 and how many must be shifted to box 4? Carry on like this all the way around. Write down a formula for the total number of matches shifted (it should be something like question 1 above) and minimize it.