Dividing Up Triangles

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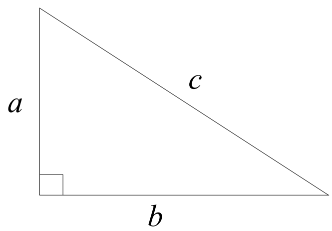
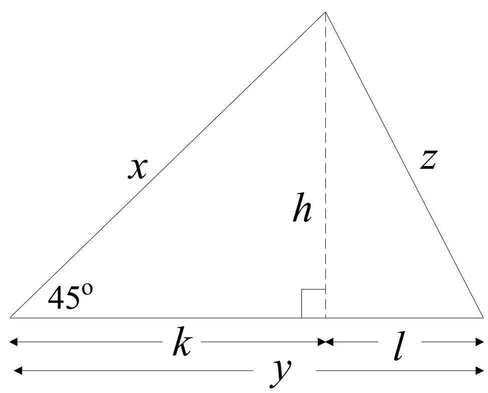
This session will be about drawing lines across a triangle which cut off a certain area. We'll begin with some useful results.



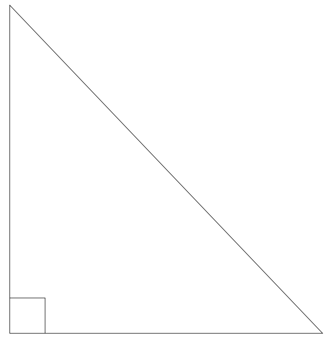
First, write down some formulas for this triangle. They must only mention a and b.

Area of this triangle =
$c^2 =$



Now write down some formulas for this triangle, this time only mentioning x and y.

Angle $\theta =$
$h =$
$k =$
Area of whole triangle =
$l =$
$z^2 =$



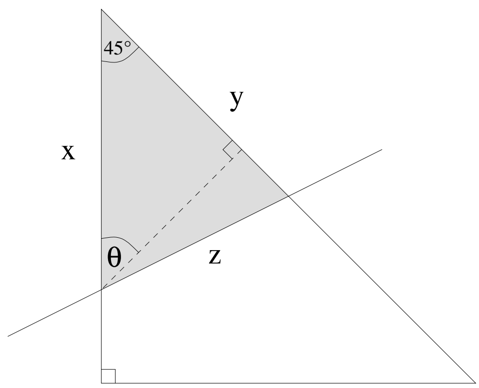
Now consider this third triangle. Calculate its area and write down the values of the angles α and ϕ.

Area of this triangle =
Angle α =
Angle ϕ =

Our next task is to find the *shortest* straight line which bisects a right-angled isosceles triangle like the one above, that is, divides it into two parts of equal area. The two parts do not have to be the same shape!

We are going to investigate two different possibilities for the line dividing up the triangle. It could either pass through the hypotenuse and one of the shorter sides, or it could pass through both of the shorter sides.

Case 1



Initially, we're going to consider when the line bisects the triangle and passes through the hypotenuse, as in the diagram above. The whole triangle has area 2, so the shaded area is 1 when the line bisects the triangle.

In this case, the shaded triangle is similar to the one at the bottom of the previous page, so we can use the formulas worked out there. You should have found that:

$$\text{Area } = \frac{xy}{2\sqrt{2}} = \frac{\sqrt{2}xy}{4}$$

So, for the shaded area of the triangle above:

$$\frac{xy}{2\sqrt{2}} = 1 \quad \text{so} \quad xy = 2\sqrt{2}$$

We want to *minimise* z, that is, make it as small as possible. Again we can use a formula worked out earlier to help us. You discovered a formula for z^2:

$$z^2 = x^2 + y^2 - \sqrt{2}xy$$

We already know that $xy = 2\sqrt{2}$, therefore:

$$\begin{aligned} z^2 &= x^2 + y^2 - \sqrt{2}xy \\ &= x^2 + y^2 - \sqrt{2}.2\sqrt{2} \\ &= x^2 + y^2 - 4 \end{aligned}$$

Although we want to minimise z, and not z^2, we can still use this formula. If the values of x and y are found that minimise $z^2 = x^2 + y^2 - 4$, then the same values of x and y will minimise $z = \sqrt{z^2}$!

We're going to look at two different approaches to minimising this formula.

Approach 1

We can use the fact that $xy = 2\sqrt{2}$ to eliminate y from the equation for z^2:

$$y = \frac{2\sqrt{2}}{x} \quad \text{so} \quad y^2 = \frac{8}{x^2}$$

$$\begin{aligned} z^2 &= x^2 + y^2 - 4 \\ &= x^2 + \frac{8}{x^2} - 4 \\ &= \left(x - \frac{a}{x}\right)^2 + b - 4 \end{aligned}$$

where a and b are some constants. You can discover the values of them by multiplying out this last equation. Write down the coefficient of $\frac{1}{x^2}$ and the constant coefficient (the term or terms on the right hand side which do not involve x, y or z).

Coefficient of $\frac{1}{x^2}$:
Constant coefficient:

By comparing the coefficient of $\frac{1}{x^2}$ which you've just worked out, and the fact that in the previous equation the coefficient of $\frac{1}{x^2}$ is 8, you should immediately be able to write down the value of a.

$a =$

Now, using the constant coefficient which you've worked out, the value of a which you've just calculated, and the fact that the constant coefficient in the previous equation is -4, you should be able to calculate b.

$-4 =$
Therefore $b =$

If you now substitute these values of a and b, you should find that:

$$z^2 = \left(x - \frac{2\sqrt{2}}{x}\right)^2 + 4\sqrt{2} - 4$$

How can we find the minimum value of this equation? Well, the squared term is *always* going to be positive, no matter what is inside the brackets, so the minimum value is going to be when the squared term is zero, ie. when $x - \frac{2\sqrt{2}}{x} = 0$.

The minimum value of z^2 is therefore:

$$z^2 = 4\sqrt{2} - 4 = 4(\sqrt{2} - 1)$$

So the minimum value of z is:

$$z = \sqrt{4(\sqrt{2} - 1)} = 2\sqrt{\sqrt{2} - 1}$$

You can work out the value of x which gives you this z by using the fact that $x - \frac{2\sqrt{2}}{x} = 0$. The value of y follows immediately from this.

Approach 2

This approach uses the following fact:

$$(x - y)^2 = (x - y)(x - y) = x^2 + y^2 - 2xy$$

Therefore:

$$x^2 + y^2 = (x - y)^2 + 2xy$$

Write down what happens if you substitute this in our formula for z^2:

$$\begin{aligned} z^2 &= x^2 + y^2 - 4 \\ &= \end{aligned}$$

We also know that $xy = 2\sqrt{2}$, so write down what happens if you substitute this in too:

$$z^2 =$$

You should have a bracket $(x - y)^2$ plus a constant or constants. The bracket $(x - y)^2 \geq 0$ so the minimum value of z^2 can be seen immediately to occur when $(x - y)^2 = 0$, ie. when $x = y$. From the constant it can be seen that this gives $z^2 = 4\sqrt{2} - 4 = 4(\sqrt{2} - 1)$ and therefore $z = 2\sqrt{\sqrt{2} - 1}$, which is the same result as obtained by the first approach.

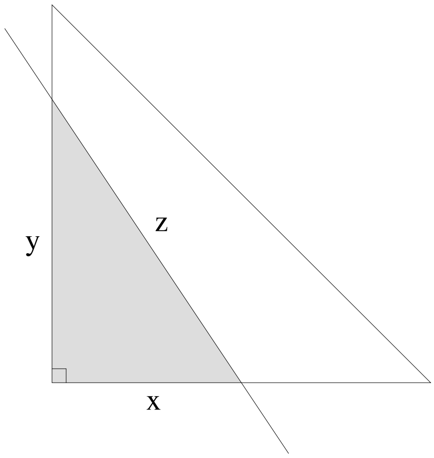
When we know that $x = y$, you can immediately find x and y from our other constraint:

$$xy = 2\sqrt{2}$$

$$\text{so } \; x^2 = y^2 =$$

$$\text{therefore } \; x = y =$$

Case 2



We haven't yet considered the case when the line bisecting the triangle passes through the two shorter sides, and not the hypotenuse, as in the diagram above. Note that the x, y and z in this diagram are different to those used previously.

The area of the large triangle is 2, and the shaded part is half of this. Write down a formula in terms of x and y for the area of the shaded part:

$$\text{so} \quad xy =$$

From Pythagoras's theorem:

$$z^2 = x^2 + y^2$$

We can use a similar trick to that used in 'Case 1, Approach 2' above, replacing $x^2 + y^2$ by $(x - y)^2 + 2xy$:

$$z^2 = (x - y)^2 + 2xy$$

But we know that $xy = 2$, so substitute this in the formula above for z^2:

$$z^2 =$$

Again, $(x - y)^2 \geq 0$, so the smallest possible value of z^2 is 4, ie. $z \geq 2$ whichever way we draw the line across the big triangle's two shorter sides.

We don't know yet, however, which approach is shorter. Is $2 > 2\sqrt{\sqrt{2} - 1}$?

If we assume this is the case, then we can treat the above inequality as an equation, doing the same operations to both sides, as long as we remember to reverse the inequality if we multiply or divide by a negative number (which in fact we don't need to do).

$$\begin{aligned} 2 &> 2\sqrt{\sqrt{2} - 1} & \\ &> & \text{(square both sides)} \\ &> & \text{(divide both sides by 4)} \\ &> & \text{(add one to both sides)} \end{aligned}$$

You should have ended up with the statement $2 > \sqrt{2}$. This is clearly true, so our original statement ($2 > 2\sqrt{\sqrt{2} - 1}$) must also have been correct.

This shows that the first case gave a shorter line, which must be the shortest line to bisect an isosceles right-angled triangle.

Some more suggestions

1. Try the same calculation with an equilateral triangle.

2. If a chord joins the point $(a, 0)$ to the point $(0, b)$ and slides, keeping its ends on the axes, so that its *length* is fixed, show that the midpoint describes a *circle*.

3. If the chord moves so that the *area* of the triangle made by the chord and the axes is constant, show that its midpoint describes a *hyperbola* (with equation of the form $xy =$ constant).

4. Take two very close chords which, as in the last example, cut off the same area. (So they will join $(a, 0)$ to $(0, b)$ and say $(a + h, 0)$ to $(0, b + k)$ where h and k are small.) Show that they intersect very nearly at the midpoint of each chord. We say that the *envelope* of the chords is the curve traced out by their midpoints, which is a hyperbola by the previous question.

5. What happens with the intersection of very close chords in Question 2?

6. (Not directly related to the above!) Show that $\sqrt{2} + \sqrt{5 - 2\sqrt{6}} = \sqrt{3}$. Of course this is supposed to be *exact*, not checked with a calculator! Can you find any other examples like this one?