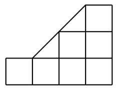
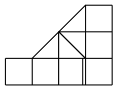
Math Club October 26 2002   SOLUTIONS

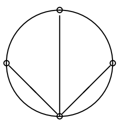
" Math Dep of Moscow University for schoolchildren
of the age 14 - 16 "

1. Cut the figure into two equal parts: 

Solution: The cut consists of two segments: 

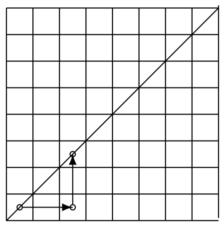
2. Twenty points are marked on a circle. Two clever boys are playing the following game: They join in turns some pair of marked points by a chord. The chords should not intersect inside the circle. The one who can't draw a new chord loses the game. Who will win (the first or the second)?

Solution: Put the points into the vertices of the regular polygon with 20 edges (actually, the solution does not depend on precise location of points, it depends only on the order of them on the circle). The first boy should join a pair of opposite nodes (leaving 9 points at each arc), that is he should make the picture symmetric. To any chord drawn by his counterpart in a semicircle he should reply by a symmetric chord at the other half of the circle (maintaining the symmetry of the picture). He always keeps the possibility to reply, so he will win.



3. Two clever girls are playing another game. Starting from the left bottom corner of the chess board they move in turns a single rook. They must shift it either to the right or upward. The one who fails to move further loses the game. Same question: who will win ?

Solution: Use similar idea to restore a "balance" (in this case, use the diagonal). The second girl will win if she replies each time by moving the rook back to the diagonal. She terminates at the right top corner.



4. Is it possible to distribute 200 identical coins between 21 people so that any two get different amount of money?

Solution: Assume they all get different amounts. Set them in increasing order. The first (smallest) number is at least zero, second is not less than 1, third is greater or equal then 3, and so on. Last (i.e., biggest) amount is at least 20. So the total amount is at least $0+1+2+\ldots+20 = 10\times(20+1) = 210$. Hence, our assumption is wrong (we have only 209 coins). So, the required distribution is not possible.

5. A boy spends a second to write down a digit. He wrote down all digits of number 2^{2002} and then of number 5^{2002}. How many days did he spend?

Solution: Let natural number a have m decimal digits, and let natural number b have m digits. This means that $10^m > a \geq 10^{m-1}$, and $10^n > b \geq 10^{n-1}$. Estimate the number of digits of the product ab. Multiply the inequalities to get:

$$10^{m+n} > ab \geq 10^{m+n-2}.$$

So the product can have either $n + m - 1$ or $n + m$ digits. Note that the product of our numbers 2^{2002} and 5^{2002} is 10^{2002} and consists of 2003 digits.

So either the total amount of digits in both numbers $(n + m)$ is either 2003 or 2004.

Actually the right answer is 2003.

Indeed, the interval $10^{n+m} > ab \geq 10^{n+m-2}$ contains only two numbers which are powers of 10 (those are 10^{n+m-1} and 10^{m+n-2}). The smaller number 10^{m+n-2} can appear only as a product of the smallest numbers 10^{m-1} and 10^{n-1} from the initial intervals, which is not the case.

So the boy spent 2003 seconds ≈ 33.4 min (roughly speaking, half an hour only).

6. A naturalist was captured by cannibals. Their chief said: You should say a statement. If it happens to be true we will eat you, if it happens to be false our pet lion will eat you. What should he do ?

Solution: He should say : " I'll be eaten by your lion." Then, neither the cannibals may eat him (otherwise, his statement becomes false, and they violate the law) nor the lion may eat him (othewise the statement becomes true, and the law is violated also).

7. Seven magicians are sitting at the round table. (You should know that a magician can be either good or evil. Besides other strange features, evils always lie, and good ones always say the truth.) It happened that each of these seven said – "One of my neighbours is good, the other is evil." Decide whether there are more good or evil magicians at the table.

Solution: Since each evil (if any) lies, he must sit either between two good ones or between two evils. *If there are two neighbouring evils* then the next one should be also evil and so on. In this case there are only evils at the table.

In other words, if there is at least one good there cannot not be two neighbouring evils.

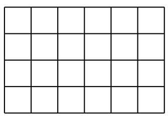
Attach numbers from 1 to 7 to all of them, say, in clockwise order. Assume 1 is good, then, one of his neighbours, say 2, is good, and, therefore, 7 is evil. Since 2 is good, then 3 is evil. Hence 4 is good, and 5 is good. Hence 6 ought to be evil, but 7 is also evil and this is not possible. So the only possibility is to have all seven evils at the table.

8. How to join all pairs of given 5 points by red and blue segments so that any triangle gets edges of different colors?

Remark: Note, that in this problem we are considering only triangles with vertices at the given points. It is not mentioned that either points are on the plane, or they form certain figure. However you may try other versions of the problem. For example, put the points in the vertices of convex pentagon (or take other configurations) and consider all triangles formed by the intersecting segments. (Show, for example, that in pentagon case the required coloring is impossible.) You can investigate all possible cases.

Solution: Considering only the triangles with vertices at the given points (the result does not depend on configuration - triangles and edges are determined by the triples or pairs of the points), it is easy to find the coloring. For example, put the points at the vertices of a convex pentagon. Colour in blue exterior edges, and colour in red all the diagonals. There are neither triangles formed by diagonals only nor triangles formed by exterior edges only.

9. Each night a swordfish cuts an edge (a thread between two nodes) of a fisherman's net (originally of size 4×6). How long the net can remain in one piece?



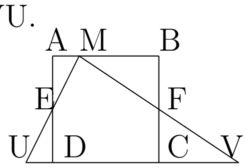
Solution: Observe, that if after a cut the net remains in one piece, the number of regions (pieces of the plane), bounded by the net decreases by 1. Really, if the swordfish cuts an edge which do not separate two different regions of the plane, then the net splits into two pieces.

Initially there were 25 regions (24 squares between the threads and an exterior infinite region). So at most during 24 nights the net can remain connected (producing finally the single region at the plane). It is easy to find the explicit sequence of these 24 cuts, for example cut successively all 6 vertical threads except the left most one in each of 4 rows.

10. Is it possible to cut a square into three parts to make up a non-equilateral triangle without right angles ?

Solution: Quite possible. Take a point M (not the middle one) on the edge AB. Draw straight lines through M and the middle points E on AD and F on BC. Mark intersection points U and V with the line CD. Triangle AME is equal to triangle DUE, while the triangle MBF is equal to triangle VCF.

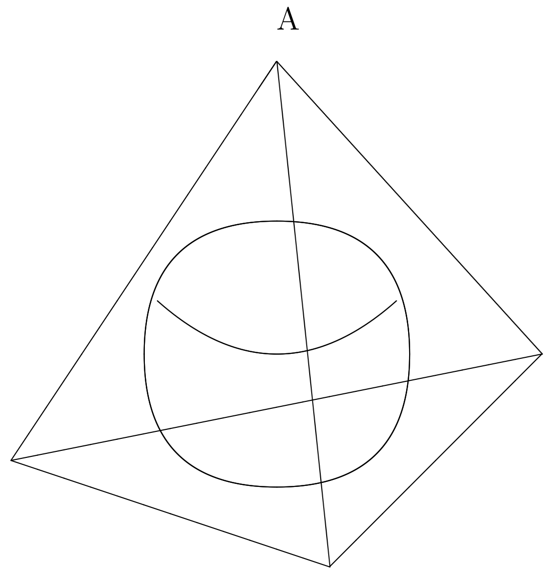
So if you take scissors and cut off the triangles AME and ABF and then put them at the place of DUE and MBF, you will get the required triangle MVU.



11. Is it possible to launch four satellites into Space so that any place on the Earth should be observable from some of them ?

Remark: We assume that satellites takes fixed positions in the Space. The answer is positive (see the solution below). One of the participants, Nathan, suggested more interesting problem: Is it possible to observe the whole surface of the Earth at any instant when the satelites are rotating along circular orbits? We give a negative answer to this assuming that the periods of rotations are equal. You may try yourself (or ask your parents) to consider other cases.

Solution: Take a regular tetrahedron such that the Earth in inscribed in it (faces are tangent to the surface of the sphere). Put satellites into the vertices. The region visible from a satellite A is bounded by a circle passing through three points of tangency of the faces adjacent to vertex A with the sphere. All four regions overlap and cover all the surface (see picture). Of course bigger tetrahedrons fits.



To answer the problem of Nathan, show at first that it is impossible to use only three satellites.

In fact, draw a plane through any two of them and a centre of our poor planet. Assume this is an Equator plane. Then North and South pole are invisible from these two satellites. The third one can't observe both poles simultaneously. Moreover, if we have four satellites but three of them are coplanar with the center of the Earth (are in the Equator plane), the fourth is unable to see both poles.

Assume now that four satellites are rotating along circular orbits with equal periods. Take any three of them (assume they are not coplanar with the Earth centre) and draw three directions from the Earth center to each of them. These satellites being taken in a certain order determine either right handed (like standard coordinate axes x, y, z in 3-space) or left handed (like y, x, z axes) triple of axes. Wait a half period time, satellites will take locations opposite to the initial ones with respect to the Earth centre. All three vectors are now changed to the opposite ones. Taken in the same order as before the satellites determine the triple of directions with the opposite orientation. Hence at some intermediate instant our three satellites become coplanar. At this very instant all four satellites fail to observe all the surface.