Advice on Hanging Pictures

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1 Introduction

When you hang a picture like a school photograph, where the width of the picture is much larger than its height, it tends to fall to a crooked position (Fig 1).

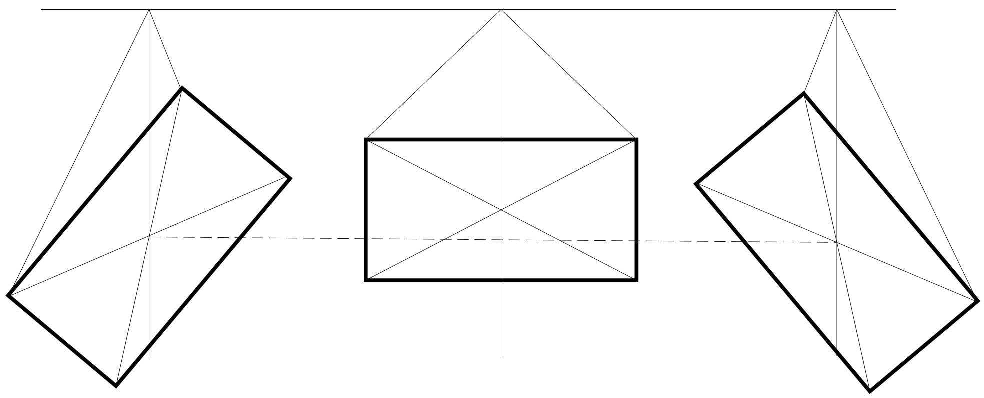

Figure 1: A school photograph hangs crooked.

But when you hang a picture like a portrait of the head teacher, where (usually) the height of the picture is larger than its width, the picture tends to hang straight. This behaviour is no reflection on the characters of the subjects involved, but follows from the (fairly) simple geometry of circles.

2 Mostly Physics

Imagine a rectangular picture $ABCD$ hung by a string which passes over a hook H and whose ends are tied to the picture at its top corners A and B.

Suppose that there is no friction between the string and the hook, and none between the picture and the wall. Suppose also that the centre of gravity G of the picture is at the intersection of the diagonals AC and BD. Then the picture will hang so that G is vertically below H.

Consider the problem of guessing where the hook might be, given only the position of the picture on the wall. (No peeping at the strings!)

If the picture is hanging straight, that is, if AB is horizontal, then the hook can be anywhere higher up than A or B on the vertical line through G.

Our main claim is that if the picture is hanging askew then *in addition* H must lie on the circle through the three points A G and B, (Fig 4, left). So in this second case there is only one possible position for H (and only one possible length for the string; a different length would cause a different hanging angle).

The rest of this article is an attempt to guide you along the proof of this claim.

Problem A: First try to prove (or simply to convince yourself) that the following two properties hold.

(1) The tensions in the two parts AH and BH of the string are equal.

(2) The lines AH and BH make equal angles with the vertical GH. That is,

$$A\hat{H}G = B\hat{H}G.$$

3 Geometry

Our claim is that property (2) implies that A, B, G and H all lie on the same circle. There are three steps.

(i) You may know—if not, ask a teacher to show you—that if a circle of centre O has a chord AG and P is a point on the circle then

$$A\hat{P}G = \frac{1}{2}A\hat{O}G.$$

There are three cases of this. If AG is a diameter then $A\hat{P}G$ is a right angle. If not, and P lies on the larger of the two circular arcs from A to G, then $A\hat{P}G$ is acute (less than 90°) and $A\hat{O}G$ is less than 180°. If P lies on the smaller of the two arcs from A to G then $A\hat{P}G$ is obtuse (between 90° and 180°) and $A\hat{O}G$ is reflex (between 180° and 360°).

(ii) It follows from (i) that as P moves along its circular arc from A to G the

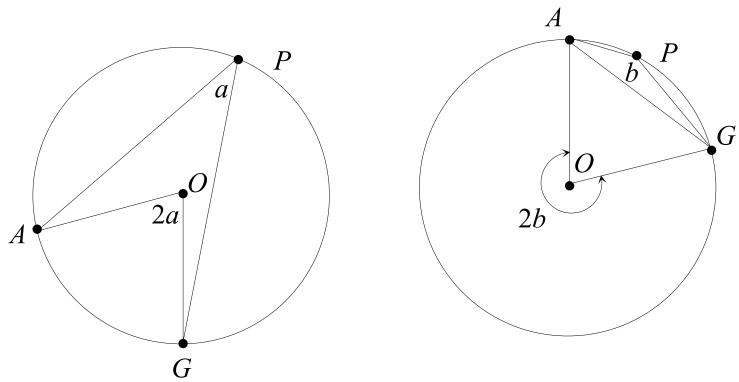

Figure 2: The angle at P is always half that at O, so long as you consider the correct angles.

angle $A\hat{P}G$ remains the same.

(iia) We shall make a small excursion here to visit a result which will be useful at the very end. If point Q lies on the arc opposite to P then the total angle $A\hat{P}G + A\hat{Q}G$ is half the sum of the angle $A\hat{O}G$ and the reflex angle $A\hat{O}G$, that is to say half of 360°, which is 180°. Thus the opposite angles P and Q of the cyclic quadrilateral $APGQ$ add up to 180°. See the left hand part of Fig 3.

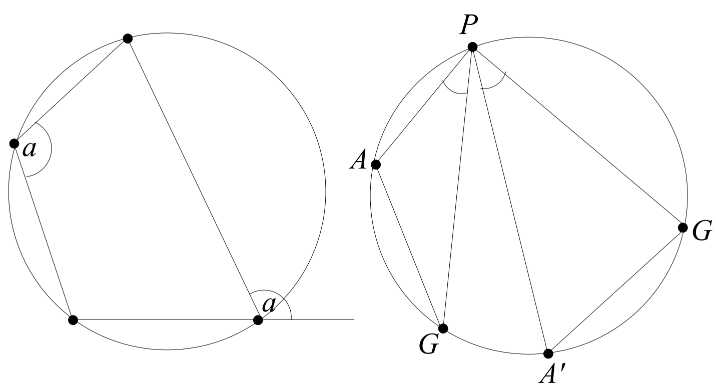

Figure 3: Left: a quadrilateral in a circle. Right: The lengths AG and $A'G'$ are equal; this makes the marked angles at P equal.

Problem B: Deduce that the angles marked a in Fig 3 (left) are equal.

(iii) We shall require a slightly different version of property (ii). Suppose we hold the point P fixed on the circle and slide the chord AG round the circle to a new position $A'G'$, keeping its length constant.

Problem C: In Fig 3, right, try to prove $A\hat{P}G = A'\hat{P}G'$.

4 The Crooked Picture

We now have the tools to hand to deconstruct the picture hanging askew. Consider the circle through the points A G and B.

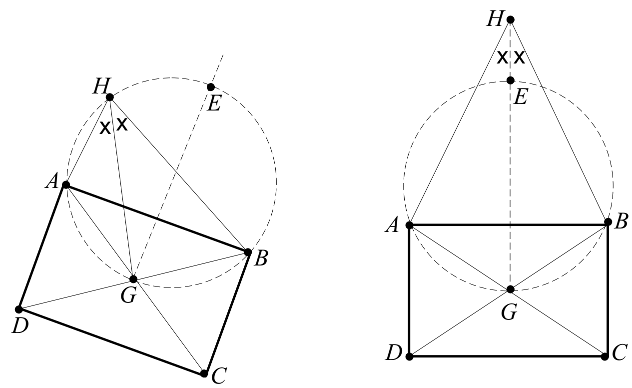

Figure 4: The circle through A, G and B plays a vital role here.

Let this circle cut the vertical line through G at the point H (Fig 4, left). The chords AG and BG have equal length since both are half-diagonals of a rectangle and so, by Problem C, the angles $A\hat{H}G$ and $B\hat{H}G$ must be equal. So H satisfies property (2) and lies vertically above G. It is therefore a possible position for the hook.

To show that H is the *only* possible position, suppose that H' were another point on GH also satisfying $A\hat{H}'G = B\hat{H}'G$. Then the triangles AHH' and BHH' would be reflections of each other in their common side HH'

and therefore congruent. Then AH would equal BH and AB would be horizontal—which it isn't!

Hence the only possible position for the hook is H.

Finally observe that from Problem B, the angle $A\hat{H}B$ between the strings is equal to the angle $B\hat{G}C$ between the diagonals of the picture. As you lengthen the string, so the hook will move along the circle to the symmetrical point E and the picture will hang straight.

If you lengthen the string further, the hook will move up the perpendicular bisector of AB and the picture will continue to hang straight.

So in the absence of friction, (and certainly in the absence of Blu-tack!) you should use enough string to make the angle between the strings less than that between the diagonals of the picture. Or use two short parallel strings, and two hooks; then the picture thinks the hook is up at infinity, and will certainly hang straight.

(Further information on this topic may be found in the article by F J Bloore and H R Morton, "Advice on Hanging Pictures", American Mathematical Monthly, Vol 92 (1985) 309-321.)