Maths Club

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University of Liverpool

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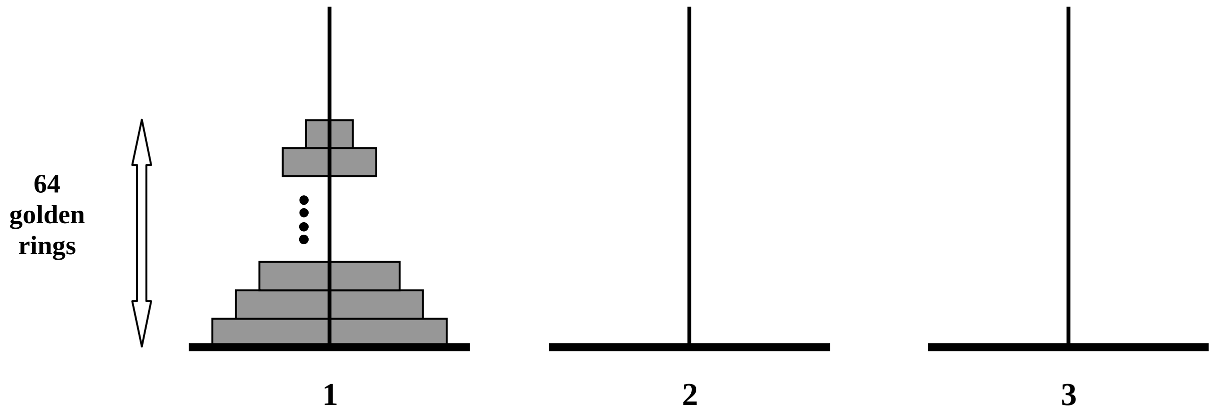
Towers of Hanoi

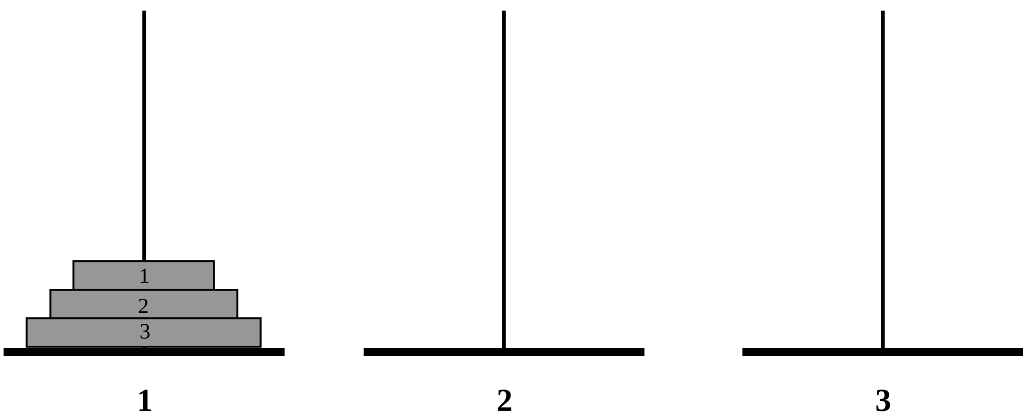
A view of Hanoi

Hanoi is the capital of Vietnam with a millennium long history. One of the most famous things associated with it is this mysterious puzzle:

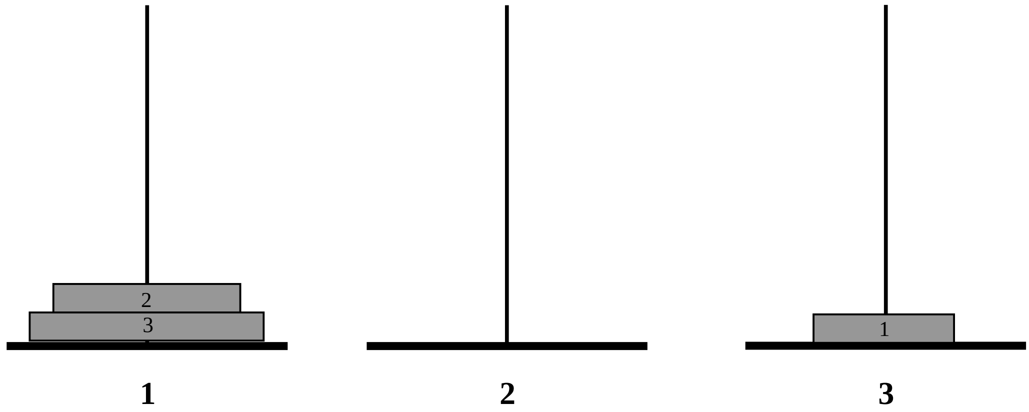
A group of monks are the keepers of three towers on which *64* golden rings sit. Originally all *64* rings were stacked on one tower with each ring smaller than the one beneath. The monks are to move the rings from this first tower to the third tower one at a time but never moving a larger ring on top of a smaller one. Once the *64* rings have all been moved, the world will come to an end.



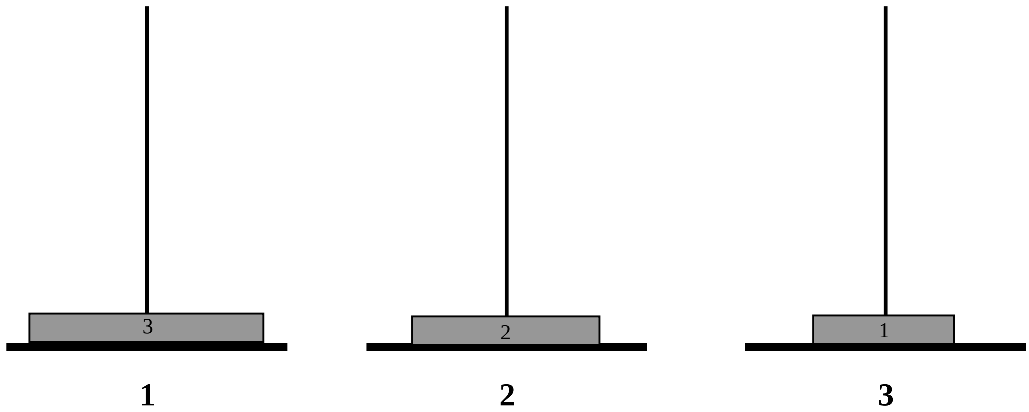
Don't be deceived by anyone who offers you millions of pounds for completing this task. Let us consider a much simpler case with 3 towers again but assume that our task is to move 3 stacked disks to tower 3 obeying the rules set by the monks.



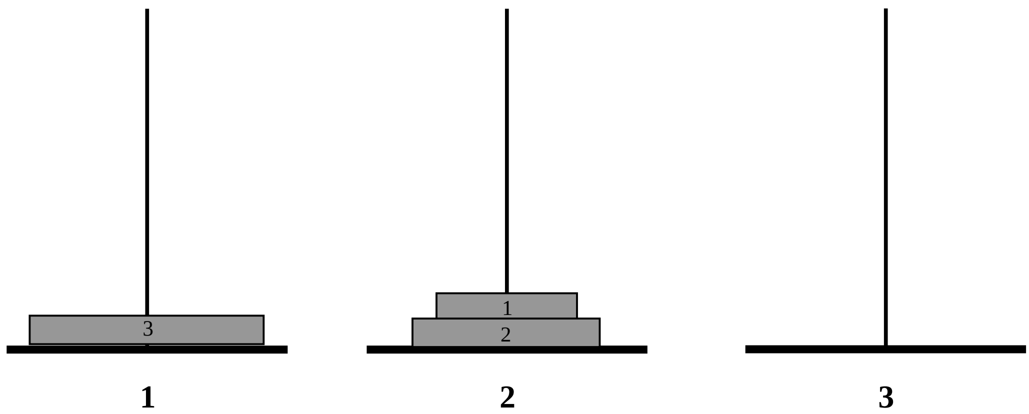
First, start with moving disk 1 to tower 3 (why not to tower 2?).



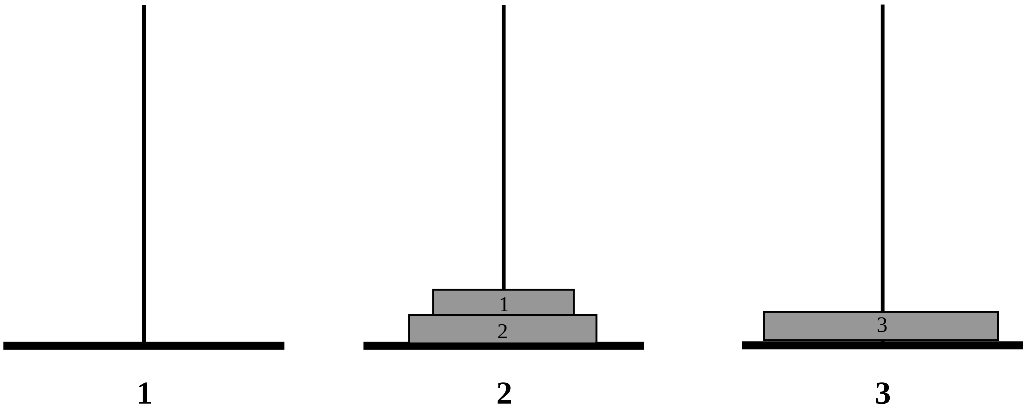
Next step is obvious as we cannot move disk 2 anywhere else but to tower 3 (it doesn't make sense to move disk 1 to tower 2!).



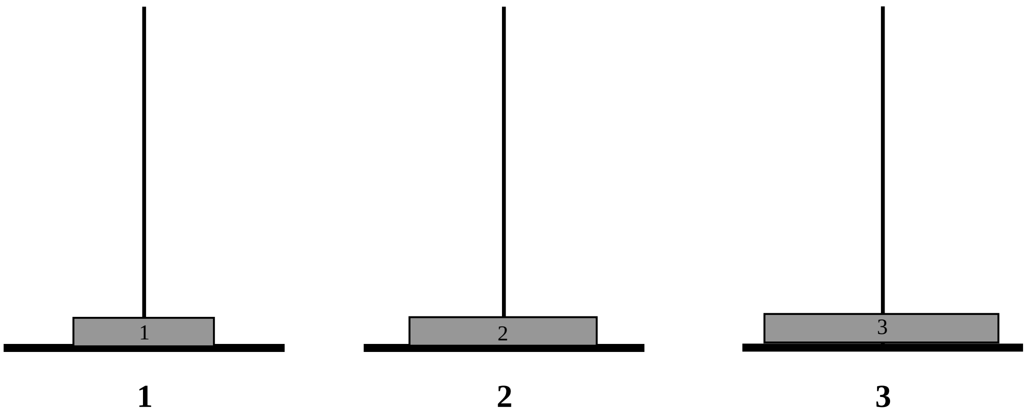
Remember that we want to move the rings from tower 1 to tower 3 (in their original order), therefore emptying tower 3 at this point would be a good idea! So, disk 1 goes to tower 2.



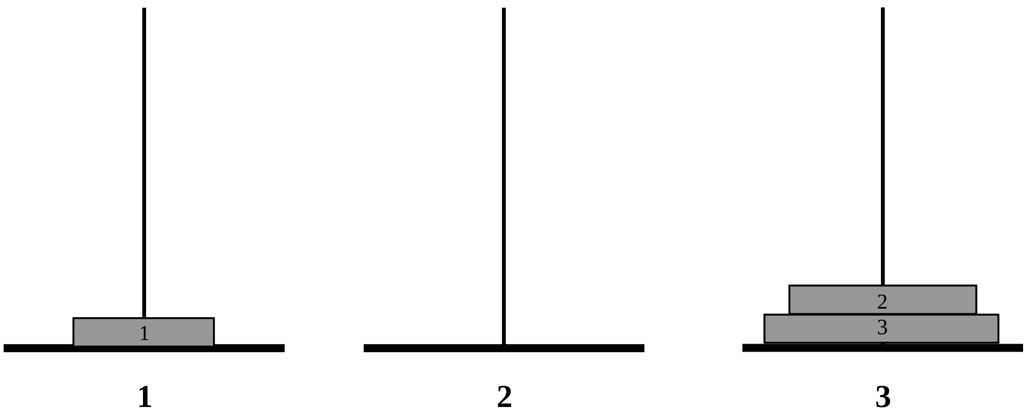
Everything looks in good shape. Move disc 3 to tower 3 now.



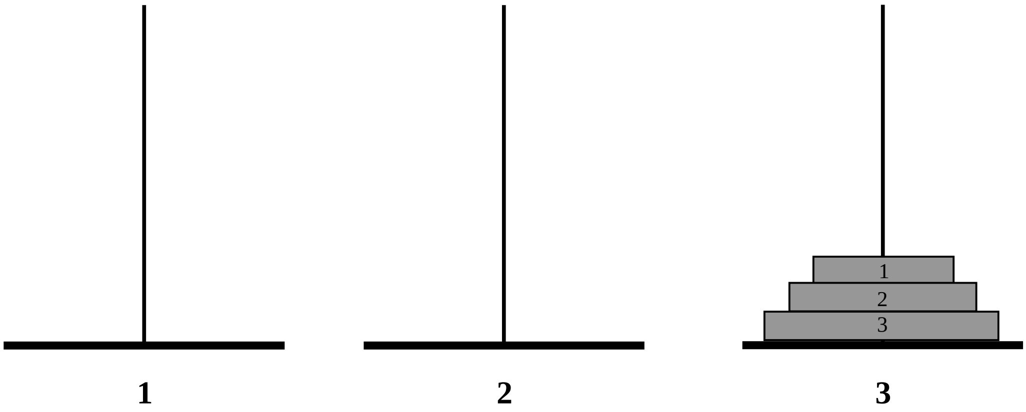
We want to put disc 2 on top of disc 3, so why not move disc 1 to tower 1?



And, as we wanted disc 2 goes to tower 3!



At last, move disc 1 to tower 3 to complete the task!



Fine, but what does it have to do with maths? Let's go back to the question we asked:

Why did we start with moving
disc 1 to tower 3 instead of tower 2?

Well, think about what would have happened if we had moved disc 1 to tower 2. Could the reason be that we wanted to move the discs faster or, in other words, with the least number of steps? Now, the question becomes more mathematical:

What is the minimum number of steps
to move 3 discs from tower 1 to tower 3?

Okay, first, have a look at the sequence:

1, 1, 2, 3, 5, 8, 13, ? , …

What comes next? Good guess.

So, you have seen the pattern. In maths, this kind of relation is called an "recurrence relation". You may say: "What about the first two numbers?" Well, these start-up values are called "initial conditions".

Let's call the numbers in the sequence

$$a_0\ ,\ a_1\ ,\ a_2\ ,\ a_3\ ,\ldots,a_{n-1}\ ,\ a_n\ ,\ldots$$

What we want to do is write down a "recurrence relation" for the above sequence as a certain equation that relates a_n to some of $a_0\ ,a_1\ ,\ldots,a_{n-1}$.

Now, the initial conditions for a sequence are just given values for a finite number of terms at the start of the sequence.

What was the question again? "*minimum number of steps*"?

If there is only 1 disc, we move it from tower 1 to tower 3 and that's it! If there is more than 1 disc (let's say "n") then we recursively use our method to move the top $n-1$ discs to tower 2 (the bottom disc is still on tower 1). Next we move the bottom disc (call it disc number n) to tower 3. Finally, we again recursively use our method to move the $n-1$ discs on tower 2 to tower 3.

So this really means that we solve the $n-1$ disc problem, then we move the bottom disc (disc number n) from tower 1 to tower 3 and finally solve the $n-1$ disc problem again. Therefore,

$$a_n = 2\,a_{n-1} + 1, \qquad n > 1$$

The initial condition is

$$a_1 = 1.$$

The question again?
"*minimum number of steps to move 3 discs?*"

After all this preparation it wouldn't be fair to solve it for 3 discs only. Let's consider n discs instead!

We know that

$$a_n = 2\,a_{n-1} + 1.$$

By looking at this equation and substituting $n-1$ wherever we see n we can also write:

$$a_{n-1} = 2\,a_{n-2} + 1.$$

Now a_n can be written in a different way:

$$a_n = 2\,(2\,a_{n-2} + 1) + 1 = 2^2\,a_{n-2} + 2 + 1$$

If we do the same thing with $a_{n-2}, a_{n-3}, \ldots$ we can write a_n differently each time:

$$
\begin{aligned}
a_n &= 2^2\, a_{n-2} + 2 + 1 \\
&= 2^2\,(2\,a_{n-3} + 1) + 2 + 1 \\
&= 2^3\, a_{n-3} + 2^2 + 2 + 1 \\
&= 2^3\,(2\,a_{n-4} + 1) + 2^2 + 2 + 1 \\
&= 2^4\, a_{n-4} + 2^3 + 2^2 + 2 + 1 \\
&. \\
&. \\
&. \\
&= 2^{n-1}\, a_{n-(n-1)} + 2^{n-2} + 2^{n-3} + \ldots + 2 + 1 \\
&= 2^{n-1}\, a_1 + 2^{n-2} + 2^{n-3} + \ldots + 2 + 1 \\
&= 2^{n-1} + 2^{n-2} + 2^{n-3} + \ldots + 2 + 1 \\
&= 2^n - 1 \qquad\qquad (*)
\end{aligned}
$$

Where did the starred line come from? (We'll discuss it during the meeting.) In mathematical terms the series above the starred one is called a "geometric series".

Well, is it clear now why we mustn't accept even a million pound offer to complete the monks' task? The minimum number of moves the task requires is:

$$2^{64} - 1 = 18,446,744,073,709,551,615$$

At this point you might say: "Okay, but how do I know that this is the minimum number of steps or how can I make sure that there isn't a way of moving all 64 discs in less than this many moves?" The answer comes from the so called "mathematical induction"! How does it work?

Let's call the number of moves required for the best solution for the n disc problem p_n. We want to show that

$$a_n = p_n, \qquad n \geq 1.$$

If $n=1$:

Remember $a_1 = 1$ (initial condition). The best solution will be just one move from tower 1 to tower 3. This means that $p_1 = 1$. Therefore, the above statement is true for $n=1$, and $a_1 = p_1 = 1$.

Now, assume that our statement is true for $n-1$. Consider the stage in the best solution to our n disc problem when we have to move the largest disc. At this point the largest disc has to be on tower 1 by itself and tower 3 has to be empty, but this means that n-1 discs have to be stacked on tower 2. We can see that this is the same as solving the n-1 disc problem first. Remember, we called the number of moves required for the best solution for the n disc problem p_n . Therefore, the number of moves required for the best solution for the n-1 disc problem will be p_{n-1} . Now, we can move the largest disc from tower 1 to tower 3 and finally, move the n-1 discs from tower 2 to tower 3 which, like before, has the best solution of p_{n-1} additional moves. This means that

$$p_n \geq 2\, p_{n-1} + 1.$$

We have already assumed that our statement is true for $n-1$ ($a_{n-1} = p_{n-1}$). Replacing p_{n-1} by a_{n-1} :

$$p_n \geq 2\, p_{n-1} + 1 = 2\, a_{n-1} + 1 = a_n.$$

The last equality in the above equation came from the recurrence relation.

No solution can take fewer moves than the best solution:

$$a_n \geq p_n.$$

Using the last two inequalities we have

$$a_n = p_n.$$

Therefore this is the best solution.

Two additional puzzles for curious brains!

1. Imagine that you've got a chess board (64 squares) and a rope. You stretch the rope and move it on the surface of the board. How many squares can you touch with the rope at most?

   Difficult?

   Why not consider a simpler case?

   Like what?

   Like a 2 x 2 chess board?

   Can you generalise it to 8 x 8 now?

2. There are an odd number of people in a crowd (well, let's say more than 4 people). You tell everyone to choose three in the crowd to share a secret and somebody tells you that that's not possible.

   Puzzled?

   Again start with a simpler case? Let's say that there are only 5 people in the crowd.

   Can you generalise it?