Sequences and Series

Formulae

The sum of the first n terms of a GP is given by

$$\begin{aligned} S_n &= a\left(\frac{1-r^n}{1-r}\right), \qquad r<1, \\ &= a\left(\frac{r^n-1}{r-1}\right), \qquad r>1, \\ &= na, \qquad r=1, \end{aligned}$$

The sum of the first n natural numbers is:

$$\begin{aligned} 1+2+3+\ldots+n &= \sum_{r=1}^{n} r \\ &= \frac{1}{2}n(n+1) \end{aligned}$$

The sum of the squares of the first n natural numbers is:

$$\begin{aligned} 1^2+2^2+3^2+\ldots+n^2 &= \sum_{r=1}^{n} r^2 \\ &= \frac{1}{6}n(n+1)(2n+1) \end{aligned}$$

The sum of the cubes of the first n natural numbers is:

$$\begin{aligned} 1^3+2^3+3^3+\ldots+n^3 &= \sum_{r=1}^{n} r^3 \\ &= \frac{1}{4}n^2(n+1)^2 \end{aligned}$$

Sum to infinity of a GP is:

$$S_\infty = \sum_{n=1}^{\infty} \frac{a}{1-r}, \qquad |r|<1.$$

Examples

1. The sum of the 2nd and 3rd terms of a GP is 12. The sum of the 3rd and 4th terms is -36. Find the first term and the common ratio.

2. A child tries to negotiate a new deal for her pocket money for the 30 days of June. She wants to be paid 1p on the 1st of the month, 2p on the 2nd of the month, and, in general, 2^{n-1}p on the nth day of the month. Calculate how much she would get, in total, if this were accepted.

3. Calculate the sum to infinity of the series $2 + \frac{1}{2} + \frac{1}{8} + \frac{1}{32} + \cdots$

4. The sum to infinity of a GP is 7 and the sum of the first two terms is $\frac{48}{7}$. Show that the common ratio, r, satisfies the equation $1 - 49r^2 = 0$. Hence find the first term of the GP with positive common ratio.

5. In a geometric progression, the sum of the firsst two terms is 9 and the third term is 12.
Find the two possible values of the common ratio r, and the corresponding values of the first term a.
Find the sum to infinity ofthe series for which $|r| < 1$.

Some challenging examples

1. Given that $x, 4, x+6$ are the sixth, seventh, and eighth terms of a geometric series and that the sum to infinity of the series exists, find the first term and the sum to infinity.

2. If the sum of the infinite geometric series

$$x^2 + \frac{x^2}{(1-x)} + \frac{x^2}{(1-x)^2} + \frac{x^2}{(1-x)^3} + \ldots$$

is 380, what are the two possible values of x?

3. The first term of a geometric series is a, where $a \neq 0$, and the second term is $a^2 - 2a$.

   a) Write down the common ration of the series, in terms of a.

   b) Find the set of values for a for which the series has a sum to infinity.

4. For $|x| < 1$ show that

$$\sum_{r=1}^{\infty} r \times x^r = \sum_{r=1}^{\infty} \left(x^r \times \sum_{r=0}^{\infty} x^s \right)$$

Hence deduce that

$$\sum_{r=1}^{\infty} r \times x^r = \frac{x}{(1-x)^2}$$

5. If $S_1, S_2, S_3, \cdots, S_p$ are the sum of infinite geometric series, whose first terms are $1, 2, 3, \cdots, p$ and whose common ratios are $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots, \frac{1}{p+1}$, respectively, prove that $S_1 + S_2 + S_3, \cdots + S_p = \frac{p}{2}(p+3)$.