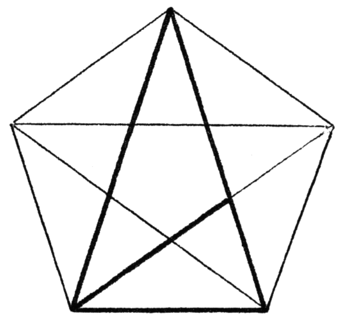
TRADITIONAL JAPANESE GEOMETRY
A Selection of problems

Sketch solutions

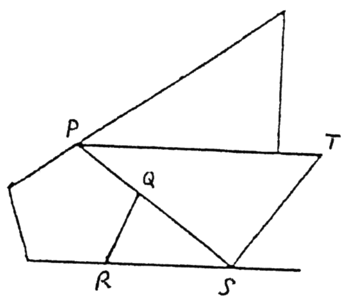
A1. AD = BD = BC = τ, so AB = AC = τ + 1. Hence, from the similar triangles ABC and BCD, $(\tau + 1) / \tau = \tau / 1$. Therefore
$\tau^2 - \tau - 1 = 0$, and so
$\tau = (1 + \sqrt{5}) / 2 = 1.618\ldots$



For a regular pentagon, the ratio of diagonal to side is τ.

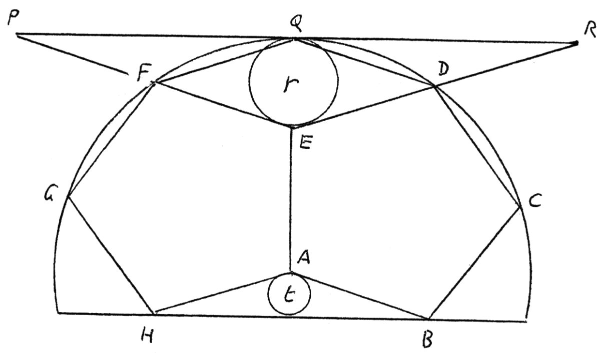
In Figure A1, cos 36° = ½ AB/BD = $\tau / 2$.

A2. In Figure A2, part of which is shown here, QR = k, so QS = τk, so

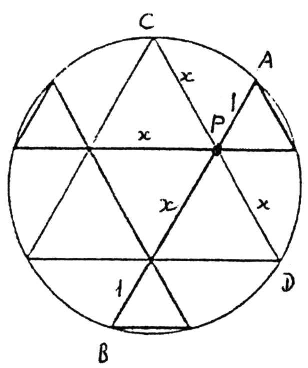


PS = k + τk + τ^2k. Also
PS/PT = cos 36° = $\tau / 2$. Hence
PT = $2\tau k = (1 + \sqrt{5})k$.

A3. All the angles in the figure are multiples of 18°. From various isosceles triangles we see that PF = FQ = FE = AH. Hence the isosceles triangles EPR and AHB are similar and EPR is twice as big as AHB.
Hence r = 2t.

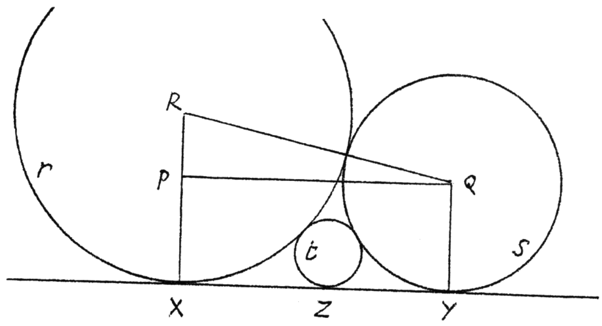


A4. In the figure, various lengths are labelled 1 and x.
PA.PB = PC.PD, so x + 1 = x^2; hence x = τ.

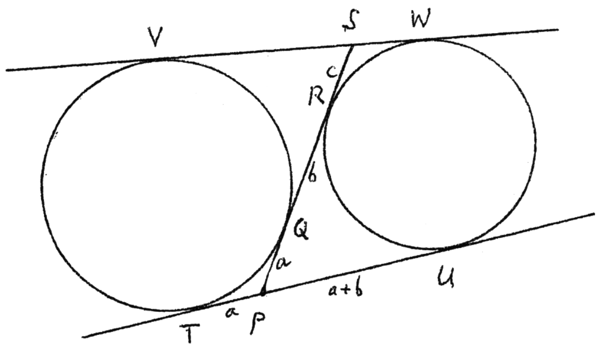


B1. $XY = PQ$, and $PQ^2 = RQ^2 - PR^2$
$= (r + s)^2 - (r - s)^2 = 4rs$.
Hence $XY = 2\sqrt{rs}$

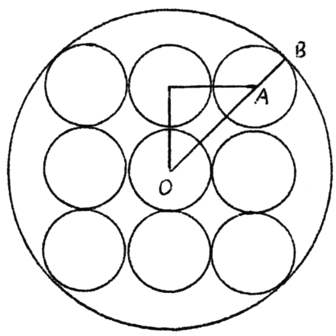
$XY = XZ + ZY$, so $2\sqrt{rs} = 2\sqrt{rt} + 2\sqrt{ts}$,
from which the result follows.



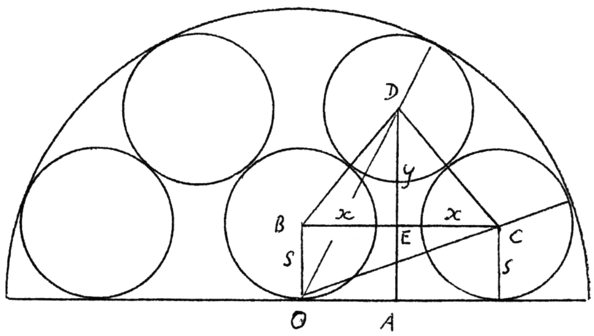
B2. In the figure, write
$PQ = a$, $QR = b$, $RS = c$. Then
$TP = a$ and $PU = PR = a + b$.
Hence $TU = 2a + b$.
Similarly $VW = 2c + b$.
But $TU = VW$. Hence $a = c$.
Hence $TU = 2a + b = PS$.



B3. $R = OA + AB = (2\sqrt{2} + 1)r$.
Hence $r = R / (2\sqrt{2} + 1) = (2\sqrt{2} - 1)R / 7$.



Suppose that five circles of radius s are packed in a semi-circle of radius R in the symmetrical manner shown. Can we prove that $s = r$?



From the right angled triangles BED, OBC and OAD,

$x^2 + y^2 = 4s^2$ (1)
$4x^2 + s^2 = (R - s)^2$ (2)
$x^2 + (s + y)^2 = (R - s)^2$ (3)

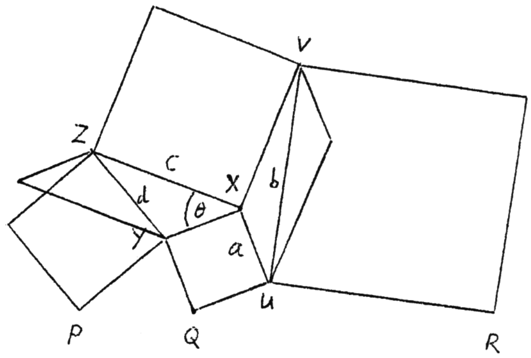
From (1) and (3), $2sy = R^2 - 2Rs - 4s^2$;
from (1) and (2), $4y^2 = 16s^2 - R^2 + 2Rs$.

We can now get two expressions for $4s^2y^2 = (2sy)^2$; equating them and simplifying we have $(R - 2s)(R^2 - 2Rs - 7s^2) = 0$.
Hence $R = (2\sqrt{2} + 1)s$, so $s = r$.

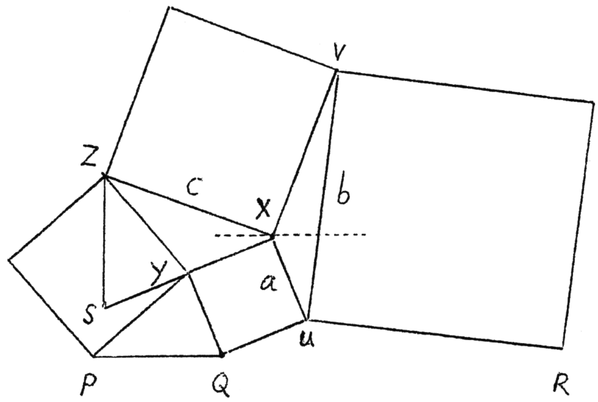
(If we put $s = 2$, *to avoid fractions, then* $x = \sqrt{7}$ *and* $y = 3$.*)*

B4. We have
$2r = (r + a) + (r + b) - (a + b) = x + y - z$.
Now if x, y and z are integers and $x^2 + y^2 = z^2$, then either x, y and z are all even or two of them are odd.
Hence 2r is even, so r is an integer.

C1. Denote the angle ZXY by θ; then the angle UXV is $180° - \theta$, and each of the triangles is half of a parallelogram with sides a and c and angles θ and $180° - \theta$.
By the cosine formula, $d^2 = a^2 + c^2 - 2ac \cos \theta$,
and $b^2 = a^2 + c^2 - 2ac \cos (180° - \theta)$
$= a^2 + c^2 + 2ac \cos \theta$;
the result follows by addition.

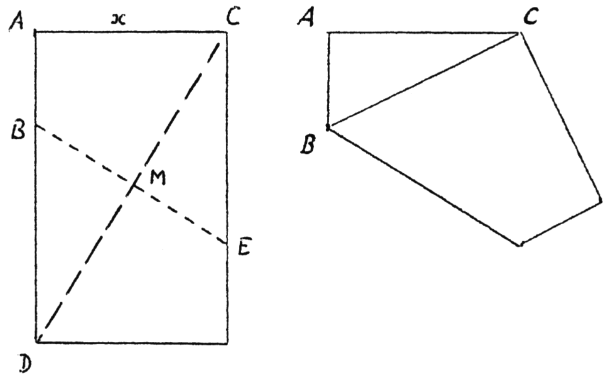


C2. Rotation through 90° clockwise transforms YQP to YSZ, so ZS $\perp$ PQ and XS = XY + YQ = 2a = c = XZ. Hence the triangle XZS is isosceles, so ZS is perpendicular to the bisector of $\angle$ZXY. Hence PQ is parallel to this bisector. Similarly QR is parallel to the bisector of $\angle$UXV. But these two bisectors are parts of the same line, so PQ and QR are parallel.



C3. IHE is a right angle (the sum of two 45° angles); also ICE is a right angle. Hence H and C both lie on the circle α on IE as diameter. The equal angles $\angle$IHC and $\angle$CHE at the point H are subtended by equal chords IC and CE of the circle α.

C4. The crease BE is the perpendicular bisector of CD. Let AD = k (fixed) and AC = x. The right angled triangles BDM and CDA are similar, so BD/DM = CD/DA.
Hence $BD = \frac{1}{2} CD^2/DA = (k^2 + x^2) / 2k$.
Hence $AB = k - BD = (k^2 - x^2) / 2k$.
The area of ABC is $x(k^2 - x^2) / 4k$,
whose minimum value occurs when $x = k/\sqrt{3}$,
as may easily be shown by calculus.



C5. The two triangles ADE and EFB, with their incircles, are similar. Hence AD/EF = r/s, so AD = tr/s. Now ED = t = ts/s, so $AE = t\sqrt{r^2 + s^2}/s$. Applying the formula of Problem B4 to the right angled triangle ADE with inradius r, we have

$$2r = tr/s + ts/s - t\sqrt{r^2 + s^2}/s$$

$$= t[r + s - \sqrt{r^2 + s^2}]/s,$$ from which we easily (?) obtain $t = r + s + \sqrt{r^2 + s^2}$.

John Rigby, 10 February 2002