Mathematics Club, The University of Liverpool
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Sums of Distances
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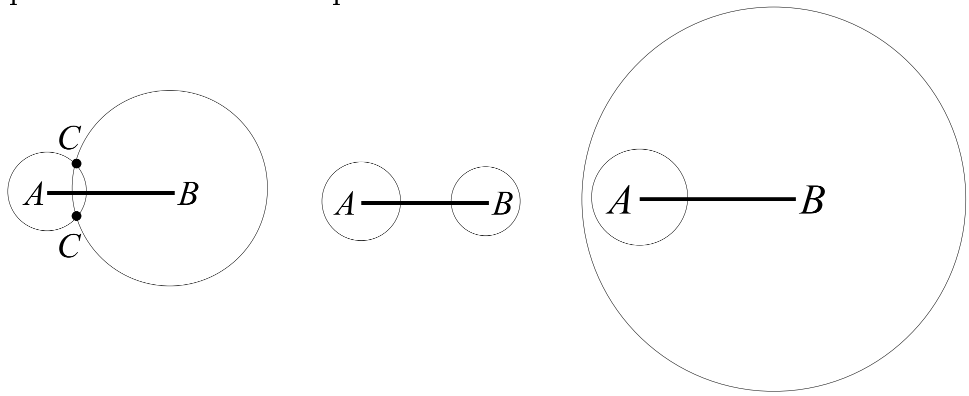
1 Triangle inequality

1.1 Geometry

Let a, b, c be three positive real numbers. Imagine trying to draw a triangle ABC having the property the three numbers are the lengths of the sides: $BC = a, AC = b, AB = c$. Laying off say AB along a horizontal line, we try to find C with $AC = b, BC = a$, and to do this we draw circles centre A with radius b and centre B with radius a. It's clear that these circles *won't meet* unless $b + a \geq c$ and that if $b + a = c$ they will meet only in one point C which is along the line AB. Also a and b must not be too different: the figure shows a couple of situations in which it will be impossible to construct the point C because the circles *don't intersect*.



The problem in the third case here is that $a - b > c$, that is $a > b + c$. In fact we need

$$a + b > c, \quad b + c > a, \quad c + a > b$$

in order for the three lengths to be the sides of a triangle:
the sum of two sides of a triangle is always greater than the third side
though we can write 'greater than or equal to' here if we don't mind a flat triangle with A, B, C in a straight line.

1.2 Algebra

It is a very interesting algebraic exercise to take three points A, B, C in the plane, with the lengths $a = BC, b = AC, c = AB$, and to prove algebraically that $a + c > b$. (The same argument of course will prove the other two inequalities.)

Here are some hints. We can choose axes so that $A = (0, 0), B = (c, 0)$ (remember $c > 0$) and $C = (p, q)$ where $q > 0$ (but we can't assume $p > 0$: why?). We have to prove

$$\sqrt{p^2 + q^2} + c > \sqrt{(p - c)^2 + q^2}.$$

Let us square both sides:

$$p^2 + q^2 + c^2 + 2c\sqrt{p^2 + q^2} > p^2 - 2pc + c^2 + q^2,$$

which, rearranging, gives

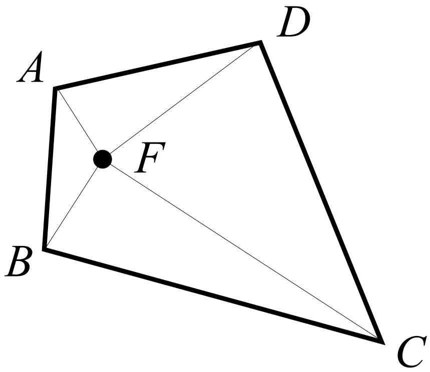
$$2c\left(\sqrt{p^2 + q^2} + p\right) > 0.$$

Can you see why this must be true? Can you, even more important, see why the argument now proves, working backwards, that $a + c > b$?

2 Four cities

2.1 Four roads

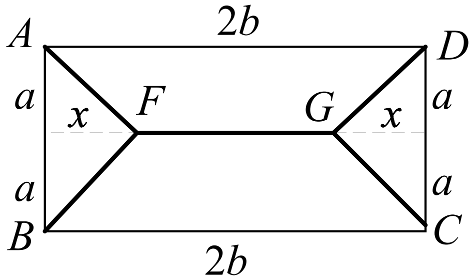
We can apply the triangle inequality to the following problem: Given four points A, B, C, D in the plane, find the point F such that the sum of the four distances $AF + BF + CF + DF$ is as small as possible. If you want to think of roads, remember that the sides AB, BC, CD, DA are *not* roads; the only roads are AF, BF, CF, DF and we want to make the total length of road as small as possible.



For simplicity we'll only look at points inside the quadrilateral, and assume that the quadrilateral is 'convex', that is all interior angles are less than 180°, as in the figure. Maybe you can use the facts that $AF + FC \geq AC$ and $BF + FD \geq BD$ to show that the point F which gives the smallest sum is in fact the intersection of the two diagonals AC, BD.

2.2 Five roads

Actually we can do better than this by using five roads instead of four. The general case here is rather difficult so let's look at a special case, that of a rectangle.



Will it always be true that the sum of the roads $AF + BF + FG + CG + DG$ will be less that the sum $AC + BD$ of the diagonals? Remember that the diagonals were the best method with only four roads. Let us determine the points F and G for which the sum of the five roads is as small as possible. For simplicity, we shall assume that FG is halfway between the parallel sides AD and BC and that F is the same distance from AB as G is from CD, as in the figure. What we want is the value of $x > 0$ that makes the total length

$$L(x) = 4\sqrt{a^2 + x^2} + 2b - 2x$$

as small as possible. It is assumed in the figure that $a \le b$.

If you know some calculus you can find the value of x very quickly by differentiation. You should find $x = a/\sqrt{3}$, which makes L take the value $L_0 = 2(a\sqrt{3} + b)$. Without calculus we can at least verify that this is indeed the smallest value of L. To do this, we have the following steps:

$$(a - x\sqrt{3})^2 \ge 0; \qquad \text{therefore } 4(a^2 + x^2) \ge 3a^2 + x^2 + 2ax\sqrt{3} = (a\sqrt{3} + x)^2,$$

and now since both sides of the inequality are positive we can take the positive square root:

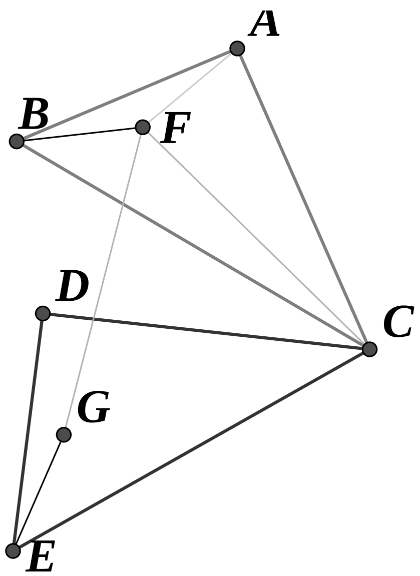
$$2\sqrt{a^2 + x^2} \ge a\sqrt{3} + x,$$

and adding $2b$ on to both sides we find that $L(x) \ge L_0$ for all x. Of course, the steps in this argument were actually discovered in reverse order, starting at the end, but it's important to realise that the steps do work in the order given here.

Note that this means that the angle FAB is 30° when the length is a minimum.

3 Three cities and three roads

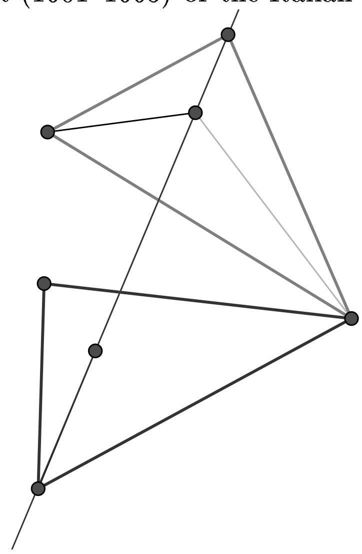
Given three points A, B, C, how can we choose a point F inside the triangle ABC so that the sum of the distances $AF + BF + CF$ is as small as possible? In the figures we will draw the triangle ABC but remember that the only roads are those from A, B, C to F. Surprisingly, this is a lot harder than the four cities and four roads problem, but there is a wonderful geometrical trick for solving this problem.



In the figure, the triangle ABC has been *rotated* about C through 60° to give DEC. Any chosen point F inside the triangle rotates too, to G say. Then $BF = EG$ since rotation keeps lengths the same. Also, because CF is rotated about C through $60°$, it follows that the triangle CFG is *equilateral*, so that $CF = FG$. Thus the sum of the distances of F from the three corners of ABC, that is $AF + BF + CF$ is actually equal to $AF + FG + GE$.

So to make $AF + BF + CF$ as small as possible, we need to make the zigzag path $AF + FG + GE$ from A to E as short as possible! This will happen when the path is *straight*, that is we want to move F around until the path $AFGE$ is a straight line as in the top figure on the next page. Remember F is the point inside the upper triangle.

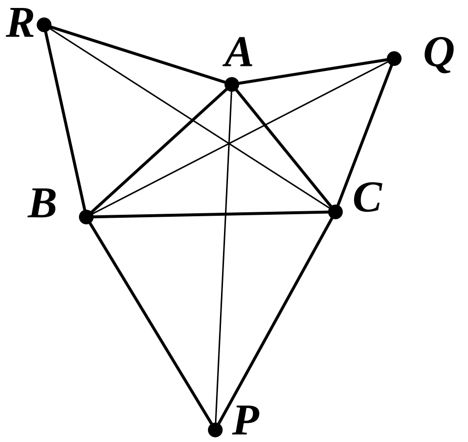
A very interesting property of this figure[1], which you might like to prove for yourself, is that with $AFGE$ a straight line the angles which AF, BF, CF make with each other are all $120°$. So the point F needs to be chosen[2] so that the angles around it, AFB, BFC, CFA are all $120°$. The resulting point F is variously named after the German Jakob Steiner (1796–1863), the Frenchman Pierre de Fermat (1601–1665) or the Italian Evangelista Torricelli (1608–1647).



The question remains: how do we find a point F inside the triangle for which all the angles AFB, BFC, CFA equal to $120°$? The figure below illustrates one possible construction. Construct equilateral triangles on the three sides of the triangle and then draw lines joining the outside vertices of these equilateral triangles to the 'opposite' vertices of the given triangle.

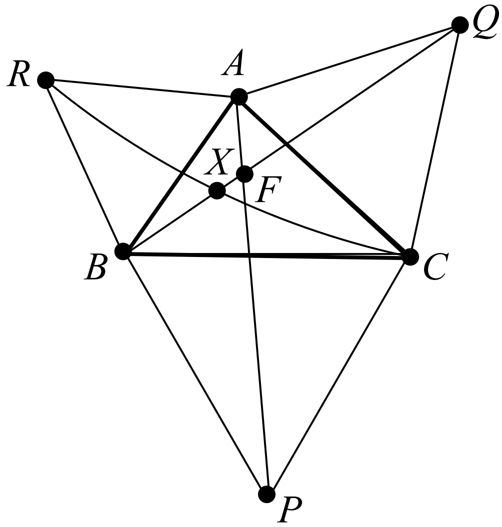
[1]For very many interesting properties of triangles and other figures, see David Wells, *The Penguin dictionary of curious and interesting geometry*, published in 1991, ISBN 0-14-011813-6. For many more advanced problems of maxima and minima, see V.M.Tikhomirov, *Stories about maxima and minima*, published in 1990 by the American Mathematical Society and the Mathematical Association of America, ISBN 0-8218-0165-1.

[2]Strictly speaking, to make F inside ABC, we need to assume that all the angles of ABC are less than $120°$.



Rotating about C through 60° takes P to B and A to Q. This proves that PA is the same length as BQ and furthermore that these lines make an angle of 60° with one-another. Similarly CR is also the same length as these two lines, and indeed any two of the lines AP, BQ, CR make an angle of 60°. This is promising, but it does not prove yet that the three lines AP, BQ, CR meet in a single point.

To do this final step suppose AP and BQ meet at F, the angle AFQ being 60° as proved above. The fact that the angles AFQ and ACQ are equal means that the four points A, F, C, Q all lie on a circle. This wonderful theorem is worth knowing! We can say 'the circumcircle of the equilateral triangle ACQ passes through F'. Likewise the circumcircle of the equilateral triangle BCP passes through F.



Suppose that the lines BQ and CR meet in X (which we want to prove is actually F: in the figure we've had to bend the line CR a bit to pretend for the time being that X is not the same as F !). Then by the same argument as that just given, the circumcircles of triangles BCP, ABR pass through X. But the circumcircle of BCP only meets BQ in B and F, so in fact X must be the point F.

This shows that in fact the three lines AP, BQ, CR do indeed meet in one point F. This is a construction for the point inside the triangle for which the sum of the distances is as small as possible.