Years, Calendars and Continued Fractions

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Liverpool University Maths Club, 26th January 2002

Everybody is familiar with π, the ratio of a circle's diameter to its circumference; a little larger than 3.14. There is no way of writing down π exactly, so we have to use approximations. We're going to start off by looking at fractional approximations to π.

Probably the most well known approxmation to π is $\frac{22}{7}$. But of course 3.14 can be expressed easily as a fraction: $\frac{314}{100} = \frac{157}{50}$. $\frac{157}{50}$ uses more numbers than $\frac{22}{7}$ but does this make it a better approximation? We know that $\frac{157}{50} = 3.14$, but using a calculator it is easy to show that $\frac{22}{7} \approx 3.14286$ which is actually slightly closer to π than 3.14.

So who "discovered" the fraction $\frac{22}{7}$? If you play about with a calculator you will see that it is the best fractional approximation to π with a single digit denominator. But it wasn't found by somebody trying hundreds of different fractions on a calculator. There are methodical methods for finding such approximations, one of which we're going to look at, called *continued fractions*.

What's the simplest possible fractional approximation to π? How about 3? We can rewrite π as:

$$\pi = 3 + 0.14159265358979\ldots$$

$0.14159265358979\ldots$ is the error in the approximation. Maybe this can be expressed as a fraction. If you take the numerator to be one you get the following equation:

$$\pi = 3 + \frac{1}{x}, \quad \text{that is} \quad \frac{1}{x} = 0.14159265358979\ldots$$

$$x = \frac{1}{0.14159265358979\ldots} = 7.0625\ldots$$

Therefore:

$$\pi = 3 + \frac{1}{7.0625\ldots}, \quad \text{so} \quad \pi \approx 3 + \frac{1}{7} = \frac{22}{7}$$

which, as we've already seen, is a good approximation to π. In fact, it's the *best* fractional approximation to π with denominator ≤ 7.

How could we go further than this to find a better approximation? Well, you can take the process one stage further as follows:

$$\pi = 3 + \frac{1}{7 + \frac{1}{y}} \quad \text{so} \quad \frac{1}{y} = 0.0625\ldots \Rightarrow y = \frac{1}{0.0625\ldots} = 15.9966\ldots$$

So a better approximation to π is given by:

$$\pi \approx 3 + \frac{1}{7+\frac{1}{15}} = 3 + \frac{1}{\frac{106}{15}} = 3 + \frac{15}{106} = \frac{333}{106} = 3.141509\ldots$$

In theory, you can repeat this process as many times as you like. In practice, it's limited by the accuracy of your calculator. Let's find one more better approximation to π:

$$\pi = 3 + \frac{1}{7+\frac{1}{15+0.9966\ldots}} = 3 + \frac{1}{7+\frac{1}{15+\frac{1}{1.00\ldots}}} \approx 3 + \frac{1}{7+\frac{1}{15+\frac{1}{1}}} = 3 + \frac{1}{7+\frac{1}{16}}$$

$$= 3 + \frac{1}{\frac{113}{16}} = 3 + \frac{16}{113} = \frac{355}{113} = 3.1415929\ldots$$

As you can see, this gives a very accurate approximation for a fraction with a relatively small denominator. This form of writing a number is called *continued fraction notation*, and can be expressed more compactly as follows:

$$\pi \approx 3 + \frac{1}{7+\frac{1}{15+\frac{1}{1+\ldots}}} = [3, 7, 15, 1, \ldots]$$

The $3, 7, 15$ and 1 in the square brackets have been read straight from the "quadruple decker" fraction on the left.

The general procedure for finding continued fraction expansions is as follows. We want to describe our number x in the form:

$$x = a + \frac{1}{y}$$

where a is the first integer in the continued fraction expansion, and y is a number (which may be irrational) on which you can repeat this process to find the next one, and so on.

The method used to find a and y is exactly the same as the method we used above. a is the largest integer less than or equal to x. This is known as the "floor function" $\lfloor x \rfloor$. If you take this away from (a positive) x, you are left with the fractional part of x, ie. the part on the right hand side of the decimal point. To find y you simply invert this, so:

$$a = \lfloor x \rfloor \quad \text{and} \quad b = \frac{1}{x - \lfloor x \rfloor} = \frac{1}{x - a}$$

This method works just as well to approximate other fractions as it does with decimals. For example, taking $x = \frac{493}{274} = 1.799270073\ldots$ we get $x = [1, 1, 3, 1, 54]$. The successive approximations are therefore:

$$1 \ , \quad 1 + \frac{1}{1} = 2 \ , \quad 1 + \frac{1}{1+\frac{1}{3}} = \frac{7}{4} = 1.75 \ , \quad 1 + \frac{1}{1+\frac{1}{3+\frac{1}{1}}} = \frac{9}{5} = 1.8$$

$$\text{and finally} \quad 1 + \frac{1}{1+\frac{1}{3+\frac{1}{1+\frac{1}{54}}}} = \frac{493}{274} \quad \text{itself.}$$

As you can see, the continued fraction expansion of this is of finite length, ie. it doesn't go on forever. The same cannot be said of the decimal expansion of many fractions, for example $\frac{1}{3} = 0.333333333333\ldots$

A *surd* is a number of the form $m + \sqrt{n}$ where both m and n are *rational numbers*, which means they can both be written as fractions with the numerator and denominator both integers. What happens when we find the continued fraction expansion for one of these? Let's try $\sqrt{2}$ ($m = 0$, $n = 2$).

$$\sqrt{2} = 1.4142136\ldots = 1 + \frac{1}{2.4142136\ldots} = 1 + \frac{1}{2 + \frac{1}{2.4142136\ldots}}$$

Clearly the continued fraction expansion is going to repeat forever as follows: $[1, 2, 2, 2, 2, 2, 2, 2, \ldots]$. Is it possible to go the other way—given something like $[3, 3, 3, 3, 3, 3, 3, 3, \ldots]$ can you find which number it represents? Well, if you call this x you can write and solve an equation as follows:

$$x = [3, 3, 3, 3, \ldots] = 3 + \frac{1}{3 + \frac{1}{3 + \frac{1}{3 + \ldots}}} = 3 + \frac{1}{x}$$

As you can see, you can replace part of the continued fraction with x itself, as it goes on forever. This equation can now be solved by multiplying through by x:

$$x^2 = 3x + 1 \quad \text{so} \quad x^2 - 3x - 1 = 0$$

$$\text{giving} \quad x = \frac{3 + \sqrt{13}}{2}$$

(the other solution: $x = \frac{3 - \sqrt{13}}{2}$ is negative so cannot be the number represented by our continued fraction expansion)

A similar but slightly different technique can be used to find what number $[3, 1, 2, 1, 2, 1, 2, 1, 2, \ldots]$ represents:

$$x = [3, 1, 2, 1, 2, 1, 2, \ldots] = 3 + \frac{1}{1 + \frac{1}{2 + \frac{1}{1 + \frac{1}{2 + \ldots}}}}$$

$$\text{so} \quad y = x - 1 = 2 + \frac{1}{1 + \frac{1}{2 + \frac{1}{1 + \frac{1}{2 + \ldots}}}} = 2 + \frac{1}{1 + \frac{1}{y}} = 2 + \frac{y}{y + 1}$$

Multiplying this out and solving gives:

$$y(y + 1) = 2(y + 1) + y$$

$$y^2 + y = 2y + 2 + y$$

$$y^2 - 2y - 2 = 0$$

$$y = \frac{2 \pm \sqrt{4-(-8)}}{2} = 1 \pm \sqrt{3}$$

$$\text{so again } y \text{ must be } \; 1+\sqrt{3} \; \text{ giving } \; x = 2+\sqrt{3}$$

Now, let's consider a problem which turns out to be related to continued fractions—the number of days in a year. A year is the length of time it takes the earth to orbit the sun, which unfortunately is not a whole number of days. In fact, it's not even a rational number of days, but is very close to 365.24219. If we had 365 or 366 days each year, the seasons would eventually get out of step with the calendar, and we'd have snow in August! What happens, of course, is that our calendar has 365 days most years, but special *leap years* with 366 days. The problem is how often we should have them.

The continued fraction expansion of 365.24219 is:

$$365.24219 = 365 + \cfrac{1}{4 + \frac{1}{7+\ldots}}$$

Clearly the first fractional approximation to 365.24219 is therefore $365\frac{1}{4}$. This would indicate that having a leap year every four years would be a good idea. This is not a brilliant approximation, however—how far out is it?

$$\begin{aligned} 365.24219 &= 365 + \frac{1}{4} - (\frac{1}{4} - 0.24219) \\ &= 365 + \frac{1}{4} - 0.00781 \\ &= 365 + \frac{1}{4} - \frac{1}{128.04\ldots} \end{aligned}$$

So having one leap year every four years would result in each year being 0.00781 days out in the long run, which is equivalent to 1 day every 128 years.

So, if we omitted the leap year every 128 years we would have a much better system. However, this would be difficult to remember and apply (off the top of your head, it's not easy to divide a given year by 128) so in Roman times Julius Caesar introduced the *Julian Calendar* with leap years every four years *except* every 100, resulting in the following approximation to a year in the long run:

$$365 + \frac{1}{4} - \frac{1}{100} = 365.24$$

This was used throughout Europe, and gradually adopted by the rest of the world until 1582, by which time (due to the inaccurate approximation and some other miscalculations) the equinoxes had slipped back by ten days. Pope Gregory XIII therefore introduced a better system: leap years every four years *except* every 100, but *do* have them every 400 (which is why 2000 *was* a leap year), giving the following approximation:

$$365 + \frac{1}{4} - \frac{1}{100} + \frac{1}{400} = 365.2425$$

To correct the slippage, 4th October 1582 was followed by 15th October 1582 in Britain. The rest of the world was slower to adopt the Gregorian Calendar, however—Turkey changed as recently as 1927, and the Greek and Russian Orthodox Churches still use the Julian calendar today.

So, how accurate is this system, and could it be improved further? In the long run, each year will be $365.2425 - 365.24219 = 0.00031$ days too long, that is it will take $\frac{1}{0.00031} \approx 3226$ years before the calendar is one day out. That's pretty good, but a better system was once devised.

The French revolution occurred in 1793. They didn't want to keep anything associated with the "old" France, and this even extended as far as the calendar! They divided the year into twelve months of thirty days, with an extra five days at the end (six in leap years). Each month was divided into three weeks, each of which lasted for ten days—this was understandably unpopular as it meant that weekends occurred less often! What interests us, however, is their rule for calculating which years should be leap years. They proposed leap years every four years *except* every 100, but *do* have them every 400 *except* every 4000. This gives the following approximation:

$$365 + \frac{1}{4} - \frac{1}{100} + \frac{1}{400} - \frac{1}{4000} = 365.24225$$

So, in the long run, each year will be $365.24225 - 365.24219 = 0.00006$ days too long, that is it will take $\frac{1}{0.00006} > 16000$ years before the calendar is one day out. Very impressive, but the calendar didn't last that long, and was abandoned just twelve years later in 1805!

References

[1] The Magical Maze, by Ian Stewart (Phoenix). This book has a section on calendars, but also a lot of interesting and accessible introductions to various mathematical topics.

[2] Calendrical Calculations, by E. M. Reingold and N. Dershowitz (Cambridge). An amazing book, dealing entirely with the mathematics of calendars, exploring in detail every major calendar that has been used throughout history, and the large number that are still in use today around the world.