Liverpool University Maths Club, Saturday, 26 January, 2002

Complex Numbers and Quadratic and Cubic Forms

Points of the real plane are presentable by *Cartesian coordinates* (x, y) or by *polar coordinates* (r, θ), where $x = r\cos\theta$, $y = r\sin\theta$). Alternatively points may be represented as complex numbers, with the real numbers identified with the points of the x-axis, while the points of the y-axis are real multiples of i, the point $(0, 1)$. Let z be the point (x, y). Then, as a complex number, $z = x + \mathrm{i}y = r(\cos\theta + \mathrm{i}\sin\theta)$. The angle θ is called the *argument* of z and the non-negative real number $|z| = r$ the *absolute value* or *modulus* (plural *moduli*) of z. The real number x is called the *real part* of z and the real number y the *imaginary part* of z.

Define the *conjugate* of z to be $\overline{z} = x - iy$. Then $z\overline{z} = x^2 + y^2 = |z|^2$.

One adds complex numbers by adding corresponding coordinates, and multiplies them by multiplying their moduli, and adding their arguments, or by using $\mathrm{i}^2 = -1$.

The complex number $\cos\theta + \mathrm{i}\sin\theta$ lies on the unit circle. Multiplication by $\cos\theta + \mathrm{i}\sin\theta$ rotates the plane through an angle θ *anti-clockwise*.

Exercise 1. Let $z = x + \mathrm{i}y$. Find the real and imaginary parts of z^2 and z^3. In each case which points have real part zero and which points have imaginary part zero? Sketch the solutions on the plane.

Exercise 2. Find the cube roots of 1. of -1, of i and of $-$i.

Exercise 3. For which real a, b, c does the quadratic form $ax^2 + 2bxy + cy^2$ have real factors? When is it plus or minus a perfect square?

Problem What can one say about cubic forms $ax^3 + 3bx^2y + 3cxy^2 + dy^3$ with real coefficients a, b, c, d?

Theorem 1 *The cubic form has three distinct real factors, two real factors with one repeated, just one real factor, or is a perfect cube according as the quadratic form*

$$(ax + by)(cx + dy) - (bx + cy)^2 = (ac - b^2)x^2 + (ad - bc)xy + (bd - c^2)y^2$$

has no real factors, is a perfect square, has two distinct real factors or is identically zero. (This quadratic form is called the *Hessian* of the cubic form.)

Exercise 4. Prove this in the special case that $a = 0$.

Exercise 5. For which real cubics is the Hessian a real multiple of xy ?

Exercise 6. For which real cubics is the Hessian a real multiple of x^2 ?

Exercise 7. For which real cubics is the Hessian a real multiple of $x^2 + y^2$?

Hint for Exercise 7:

When $ad - bc = 0$, with $a \neq 0$, then, if we let $c = \lambda a$, then $d = \lambda b$.

λ is the Greek letter *lambda*.

Exercise 8. For which real cubics is the Hessian identically zero?

Notes

Just to remind you, the circle, centre the origin of the coordinate plane, and with radius 1 is known as the *unit circle*. It has equation $x^2 + y^2 = 1$, as follows from Pythagoras' Theorem. The points of this circle represent complex numbers of modulus 1. The product of any two complex numbers on this circle is another point of the circle, with argument the sum of the arguments of the original two numbers.

The two square roots of any non-zero complex number lie on opposite sides of the origin and an equal distance from it, one being minus the other.

The three cube roots of any non-zero complex number lie at the vertices of an equilateral triangle, whose vertices are all at the same distance from the origin. So once you have found one of them you have found them all!

Some further useful information:

For any x, y,

$$x^2 - y^2 = (x - y)(x + y),$$

$$x^2 + y^2 \text{ does not have real factors,}$$

$$x^3 - y^3 = (x - y)(x^2 + xy + y^2),$$

$$x^2 + xy + y^2 = (x + \tfrac{1}{2}y)^2 + \tfrac{3}{4}y^2, \text{ so does not factorise further,}$$

$$x^3 + y^3 = (x + y)(x^2 - xy + y^2),$$

$$x^2 - xy + y^2 = (x - \tfrac{1}{2}y)^2 + \tfrac{3}{4}y^2, \text{ so does not factorise further.}$$

Also, to help you in Exercise 3, note that

$$a(ax^2 + 2bxy + cy^2) = (ax + by)^2 + (ac - b^2)y^2.$$

This is an example of the technique known as *completing the square*. The expression $ac - b^2$ is called the *discriminant* of the form $ax^2 + 2bxy + cy^2$.

In this question suppose therefore to begin with that $a \neq 0$, and deal with the special case that $a = 0$ afterwards.

What is important in practice is whether the discriminant is $> 0, = 0$ or < 0.

Solutions

Exercise 1. The real part of z^2 is $x^2 - y^2$ and the imaginary part is $2xy$. The first is zero along two lines perpendicular to each other and inclined at 45 degrees to the axes. The second is zero along the axes themselves.

The real part of z^3 is $x^3 - 3xy^2$ and the imaginary part is $3x^2y - y^3$.The first is zero along three lines, one the y-axis and the other two inclined to it and to each other at 60 degrees. The second is zero along three lines, one the x-axis and the other two inclined to it and to each other at 60 degrees.

Exercise 2. The solution in each case consists of three points on the unit circle forming the vertices of an equilateral triangle, one of the vertices in the first case being 1, in the second case -1, in the third case $-\mathrm{i}$ and in the fourth i.

Exercise 3. By *completing the square* one finds that $ax^2+2bxy+cy^2$ has distinct real factors when $ac - b^2 < 0$ (for example when $a = c = 0$, but $b \neq 0$) and is plus or minus a perfect square when $ac-b^2 = 0$. When $ac-b^2 > 0$ (for example when $b = 0$ and $ac > 0$) then it has no real factors. The quadratic form is then said to be, respectively, *hyperbolic*, *parabolic* or *elliptic*. The number $ac - b^2$ is the *discriminant* of the quadratic form.

Exercise 4. When $a = 0$ the cubic has y as one real factor, and the remaining quadratic factor has discriminant $\frac{3}{4}(4bd - 3c^2)$, while the Hessian has discriminant $\frac{1}{4}b^2(3c^2 - 4bd)$. The theorem follows at once in this case.

Exercise 5. All cubic forms of the form $ax^3 + dy^3, ad \neq 0$. For if $ac = b^2$ and $bd = c^2$, with $ad \neq bc$, then $abd = ac^2 = b^2c$, implying that $b = 0$, and therefore also that $c = 0$, since $ad \neq bc$. This form has just one real factor, namely $\alpha x + \delta y$, where α and δ are the real cubic roots of a and d.

Exercise 6. All cubic forms of the form $ax^3 + 3bx^2y$, with $b \neq 0$. Note that this has x^2 as a factor!

Exercise 7. All cubic forms of the form $a(x^3 - 3xy^2) - d(3x^2y - y^3)$. In fact it turns out that the number λ of the hint has to be equal to -1. This form has three real factors, obvious if either a or $d = 0$. However it is in any case the real part of the complex number $(a+\mathrm{i}b)(x+\mathrm{i}y)^3$, that is the real part of the cube of $x' + \mathrm{i}y'$, where $x' + \mathrm{i}y'$ is the product of one of the complex cube roots of $a+\mathrm{i}b$ and $x + \mathrm{i}y$. This, being equal to $x'(x'^2 - 3y'^2)$, also has three real factors.

Exercise 8. All cubic forms that are perfect cubes.

Remark: A real cubic form is said to be *hyperbolic*, *parabolic* or *elliptic*, according as its Hessian quadratic form is hyperbolic, parabolic or elliptic. The only remaining case is when the cubic is a perfect cube, in which case the Hessian is identically zero.