Department of Mathematical Sciences
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From Maxima and Minima to Surface Shape
Christmas Lecture, 2001

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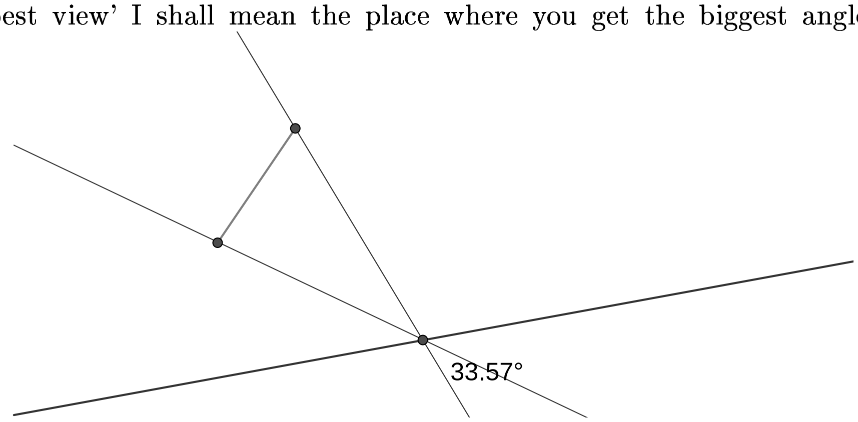
I shall look at three situations where 'maxima' (biggest values) and 'minima' (smallest values) are to be determined. The first one involves angles, the second one involves lengths and the third one involves both in a surprising way. Largest and smallest values are of enormous importance in applications of mathematics, from companies wanting to maximize their profits to scientists trying to find the most efficient way of doing an experiment or computer scientists trying to write the most efficient code for a particular computer program.

1 The stately home problem

A party of tourists are driving on a straight road past a stately home which has a beautiful front which they want to photograph. Where should they stop so as to get the best view?

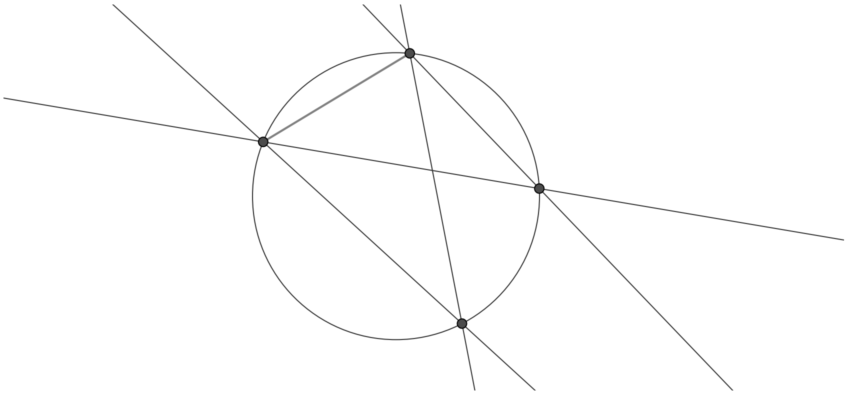
By the 'best view' I shall mean the place where you get the biggest angle of view. In



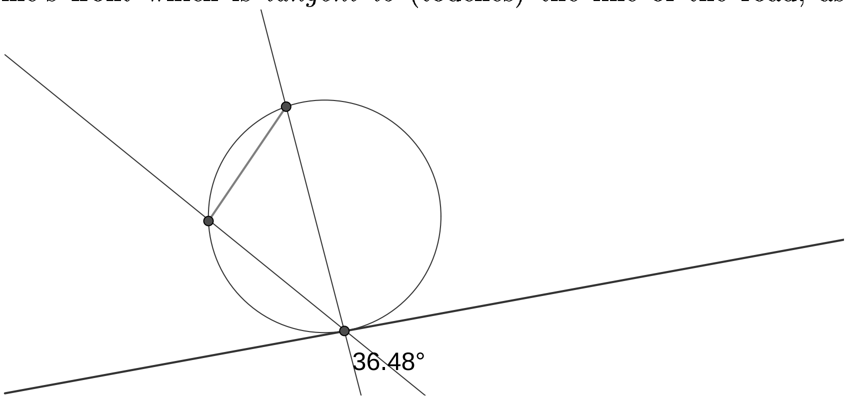
the figure[1] the angle is 33.57° but maybe it will increase if the bus moves left or right. If the bus is very far to the left the angle is small, it rises to a *maximum* as the coach drives past the stately home, and then decreases to a small angle as the coach drives off to the right.

It turns out to be very useful to consider a *circle* through the two ends of the

[1]Several of the diagrams here, and several of the computer demonstrations in the lecture, were produced with *Cinderella*, interactive geometry software, by Jürgen Richter-Gebert and Ulrich H. Kortenkamp, Springer 1999. ISBN 3-540-14719-5, http://www.cinderella.de



stately home's front. This is because of a wonderful theorem which says that if the tourists were at any point on a particular circle of this kind, they would get the *same* angle of view. Furthermore, the bigger the circle the smaller the angle. Now unfortunately the tourists are not on a circle but on a certain straight line! Consider a circle through the ends of the stately home's front which is *tangent to* (touches) the line of the road, as in the figure.



If the coach moves slightly to the left or right of this point of tangency then the circle gets bigger and the angle gets smaller. So this position represents a *maximum* of angle.

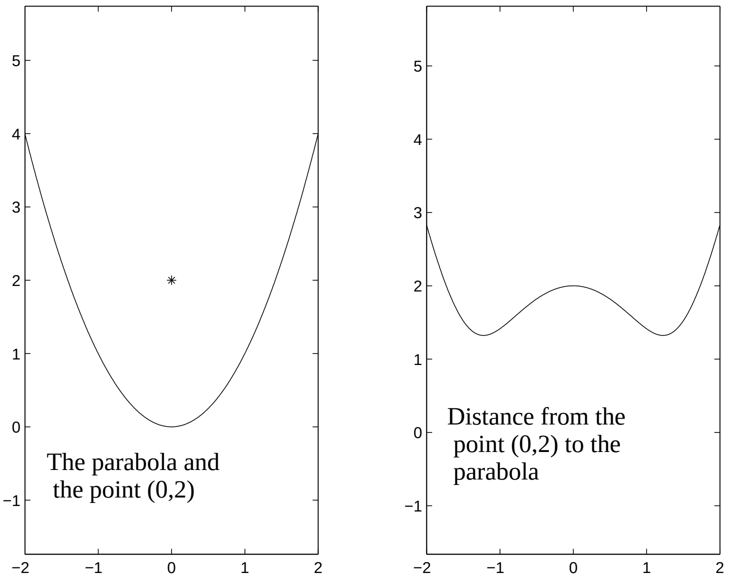
Question Given the two ends A, B of the stately home's front, and the line of the road, how would you find, or construct, the point just described: where a circle through A and B is tangent to the road?

Note also that the road need not be straight: curvy roads where a circle through A and B is tangent to the road also give a *local* maximum or minimum of angle.

2 Distances to curves

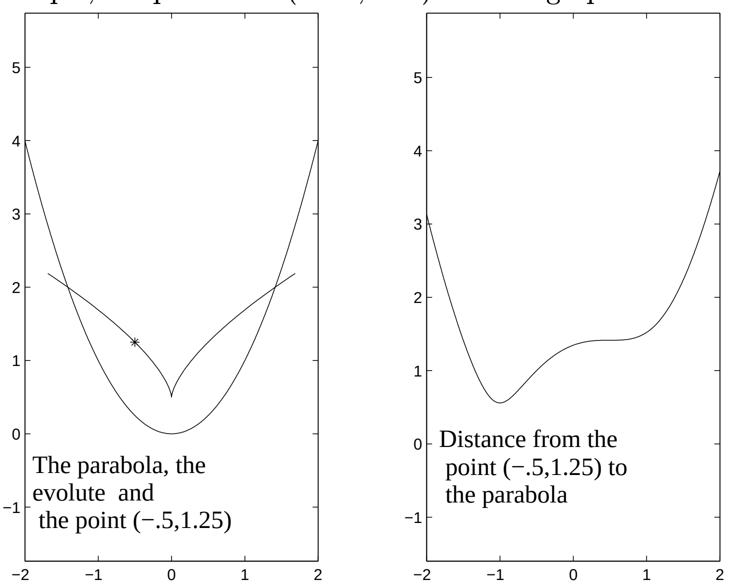
Let's take the example of a parabola, with equation $y = x^2$, and a point P which is fixed. Measure distances from P to the points of the parabola, and draw a graph of this distance, as in the figure[2]. Here the point P is $(0, 2)$ and the distance function has two minima at the same level.

[2]This and the next figure were produced using MATLAB, http://www.mathworks.com



This will always happen when the point P is on the y axis, provided it is above the point $(0, \frac{1}{2})$. Below that point there is only one minimum (see below).

In the second example, the point P is $(-0.5, 1.25)$ and the graph of distance has a *horizontal*



inflexion. In fact there is a special curve, shown in the left-hand figure, and whenever P sits above this curve the distance function has two minima (as in the first example $P = (0, 2)$), and whenever P is below this curve the distance has one minimum. When P is *on* the special curve, as in the second example above, the distance function has an inflexion and one minimum. The special curve is called the *evolute* of the parabola. If you know some calculus you might be able to show that the equation of the evolute is $27x^2 = 2(2y - 1)^3$.

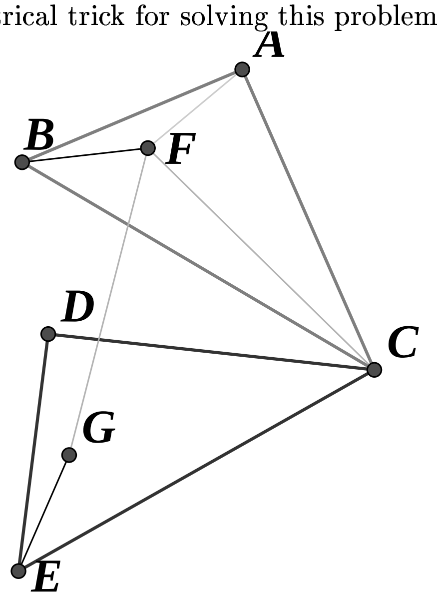
Both the ideas above are very important: points of a curve C (in the example C is a parabola) from which the distance has two equal minima are called *medial axis* points, and the medial axis is a very important tool in an application of mathematics called computer vision, where data about the real world is extracted from photographic or video images. Another way of finding medial axis points is to find centres of circles which are tangent to the curve C in two points.

The same procedure can apply to a *surface* S, and there we use spheres instead of circles to measure the shape of the surface. There are analogues of both the ideas above: the special curve giving inflexions of the distance function becomes the 'focal surface' of S, and the centres of spheres tangent to S in two points trace out a set which is still called the 'medial axis' of S.

3 Three cities and three roads

Given three points ('cities') A, B, C, how can we choose a point F inside the triangle ABC so that the sum of the distances $AF + BF + CF$ is as small as possible? (These represent roads of minimum length connecting the cities.) In the figures we will draw the triangle ABC but remember that the only roads are those from A, B, C to F.

There is a wonderful geometrical trick for solving this problem.



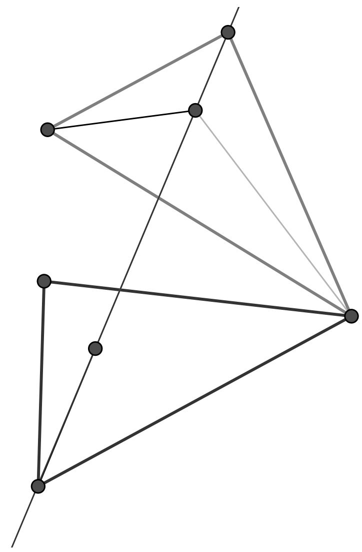
In the figure, the triangle ABC has been *rotated* about C through $60°$ to give DEC. Any chosen point F inside the triangle rotates too, to G say. Then $BF = EG$ since rotation keeps lengths the same. Also, because CF is rotated about C through $60°$, it follows that the triangle CFG is *equilateral*, so that $CF = FG$. Thus the sum of the distances of F from the three corners of ABC, that is $AF + BF + CF$ is actually equal to $AF + FG + GE$.

So to make $AF + BF + CF$ as small as possible, we need to make the zigzag path $AF + FG + GE$ from A to E as short as possible! This will happen when the path is *straight*, that is we want to move F around until the path $AFGE$ is a straight line as in the top figure on the next page. Remember F is the point inside the upper triangle.

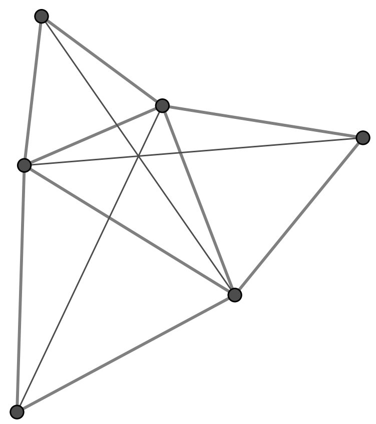
A very interesting property of this figure[3], which you might like to prove for yourself, is that with $AFGE$ a straight line the angles which AF, BF, CF make with each other are all $120°$. So the point F needs to be chosen[4] so that the angles around it, AFB, BFC, CFA are all $120°$. The resulting point F is variously named after the German Jakob Steiner (1796–1863), the Frenchman Pierre de Fermat (1601–1665) or the Italian Evangelista Torricelli (1608–1647).

[3] For very many interesting properties of triangles and other figures, see David Wells, *The Penguin dictionary of curious and interesting geometry*, published in 1991, ISBN 0-14-011813-6. For many more advanced problems of maxima and minima, see V.M.Tikhomirov, *Stories about maxima and minima*, published in 1990 by the American Mathematical Society and the Mathematical Association of America, ISBN 0-8218-0165-1.

[4] Strictly speaking, to make F inside ABC, we need to assume that all the angles of ABC are less than $120°$.



The question remains: how do we find a point F inside the triangle for which all the angles AFB, BFC, CFA equal to $120°$? The figure below illustrates one possible construction. Construct equilateral triangles on the three sides of the triangle and then draw lines joining the outside vertices of these equilateral triangles to the 'opposite' vertices of the given triangle.



Then
(i) the three lines meet together in one point;
(ii) the angles around the central point are all $60°$ so this central point is the one which we want;
(ii) all these three lines have the same length.
If you can show why any of these three statements is true, then send your solution to Professor Peter Giblin, Department of Mathematical Sciences, The University of Liverpool, Liverpool L69 3BX. There will be a **prize** for the best solution.

Of course this problem of the three points A, B, C can be generalised a lot. What about four points A, B, C, D? Where is the best position for F so that $AF + BF + CF + DF$ is as small as possible? (The answer is to choose F to be the intersection of AC and BD.) In fact to make roads of the shortest total length connecting A, B, C and D it turns out to be best not to choose a single point inside $ABCD$ but to make a more complex network with *two* points F_1, F_2 inside $ABCD$. These are joined to each other, A, B are joined to F_1 and C, D joined to F_2. Try taking $ABCD$ to be a rectangle and see if you can find the best positions for F_1 and F_2.