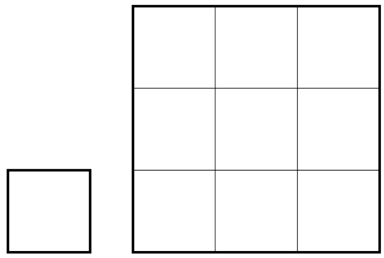
Liverpool University Maths Club, 27 October 2001
Snowflakes and other fractals

Peter Giblin

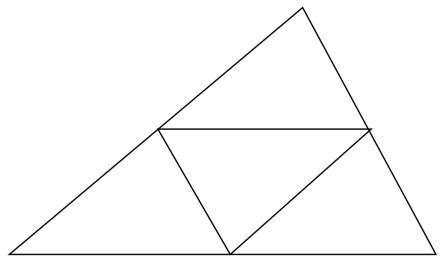
Nested Triangles

We'll calculate the lengths and areas of some complicated figures, but step by step so you can see what is happening. We'll also look at some miraculous methods for doing 'all the steps at once'.

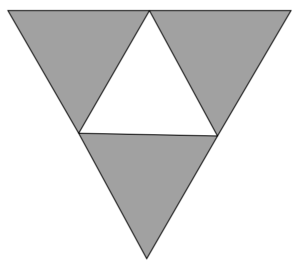
One crucial idea is that of *similarity.* When one figure is twice as big as another, the *perimeter* of the larger figure is twice as big as that of the smaller one but the *area* is four times as big. When one figure has all its dimensions multiplied by n then the perimeter is multiplied by n but the area by n^2.



In the figure, the left-hand square has a side which is one-third of that of the right-hand square. The perimeter of the small square is one-third that of the large square but the area is one-ninth.



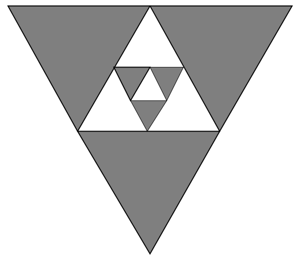
In this figure, the inner triangle passes through the midpoints of the sides of the outer triangle. This makes a figure composed of four congruent small triangles. So the area of the big triangle is four times that of the small triangle, while its side is only two times that of the small triangle.



Now let's start with an equilateral triangle. The area will be referred to as T. *Step 1* is to join up the middle points of the sides and shade in three of the small triangles, as in the figure on the right.

The total shaded area is............................of T.

For *Step 2* we continue to join midpoints of sides to obtain this new picture. The shape made up of three new shaded triangles has edge-length



a fraction....................of that of the shape made of three shaded triangles at Step 1.

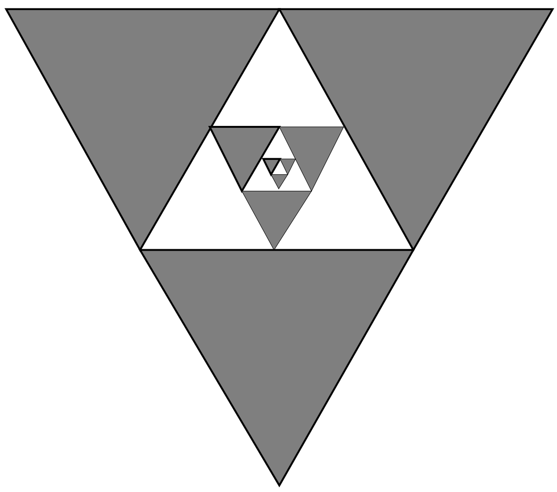
So the three new shaded triangles at Step 2 have a

total area which is a fraction......................of the area of the shaded triangles at Step 1,

that is....................of T.

The total shaded area is now ....of T. This can be written $\frac{3}{4}$(........ + ...).

Of course, we can continue in this way, though the changes become very hard to see! Let's draw the picture a bit bigger so that it is easier to see the small triangles introduced in the middle, in *Step 3*.



We proceed as before. The figure of three tiny shaded triangles has edge-length a fraction

........................of the edge-length of the figure of three new shaded triangles at Step 2. So

their total area is a fraction...................of the area added in Step 2, that is.................of T.

The total shaded area is now $\frac{3}{4}$(................. + ... + ...) = ... of T.

This is almost *name a simple fraction* ....of T.

If we go on making smaller and smaller triangles, the shaded area will get closer and closer to this fraction of T. Here is a miraculous way of seeing this.

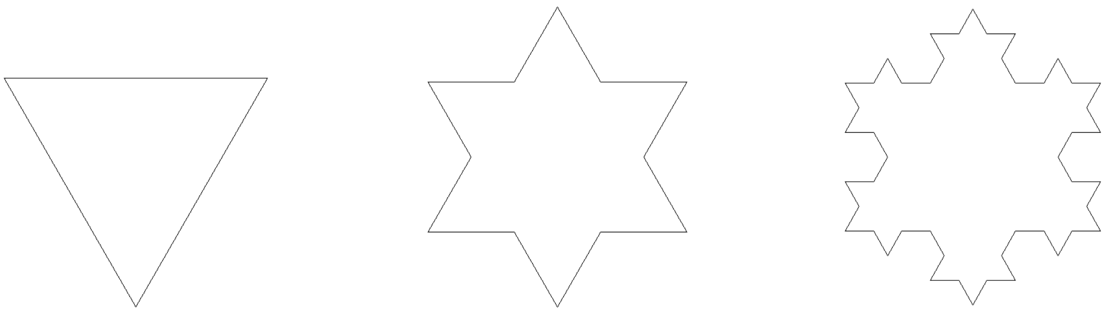
Let A be the total shaded area, when we go on filling in tinier and tinier triangles. Now look at only the shaded triangles introduced from *Step 2* onwards. These are simply a *scaled down* version of the whole, in fact a one-quarter size version of the whole. The area of the shaded triangles from *Step 2* onwards is therefore

a fraction........................of A. Also *Step 1* introduced shaded triangles with area $\frac{3}{4}T$.

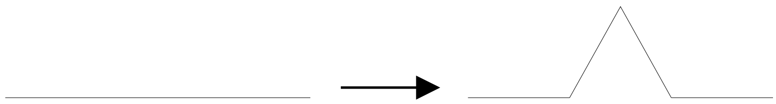
Therefore $A = \frac{3}{4}T +$ ....A. Solving this we get $A =$ ....T.

Snowflakes

Here is a picture of the first three steps of drawing a snowflake curve.



As you can see, each step involved replacing a straight edge by four edges, each one-third the length of the original edge:



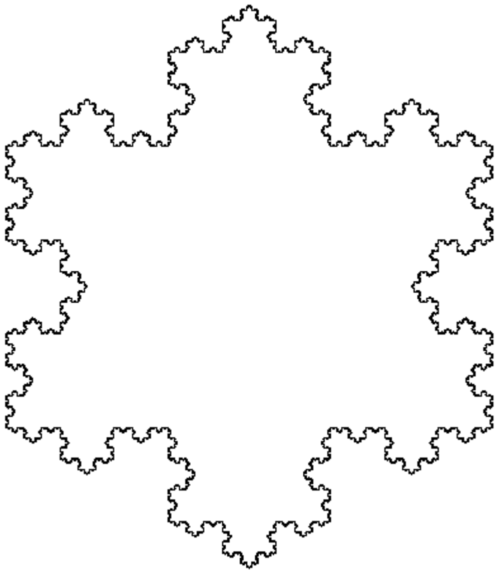
Notice that a new triangular area is added to the snowflake *every time* a line is broken in this way. Here is a table to work out what is happening here to the lengths and areas. We use T as before for the area of the original triangle. The edge-length of the original triangle is taken to be 1. Triangles added and area added mean what happens in passing from one step to the next.

Level	Number of sides	Length of each	Total length	Number of triangles added	Area of each	Area added	Total area
0	3	1	3	0	–	–	T
1	12	$\frac{1}{3}$		3			
2							
3							
4							

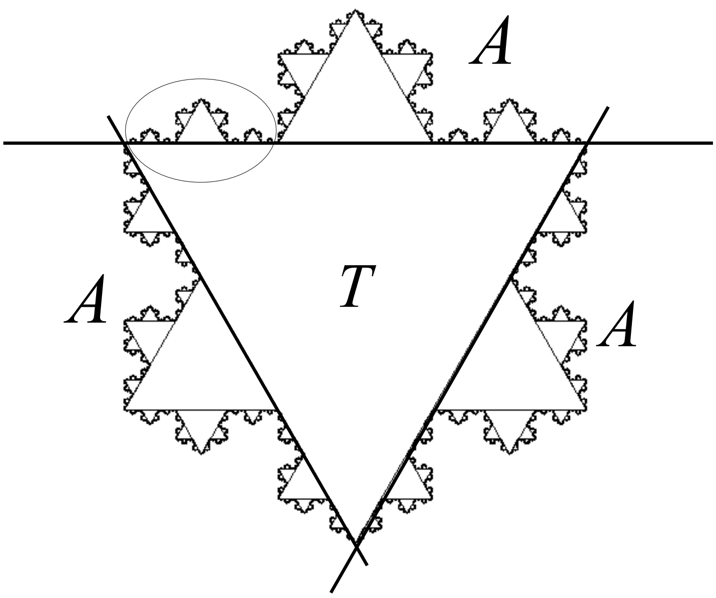
Any guesses about what is happening to the total length? The total area?

In passing from one step to the next what is happening to the length?

Here is a picture of the snowflake after several steps:



If we take many steps the edge of the snowflake becomes extremely frilly! Here is a miraculous method, similar to the one given above, for calculating what we expect the eventual area to be, after many steps.



Let A be the area inside the frilly snowflake curve and outside one edge of the original big triangle—see the figure, for the moment ignoring the oval shape. The total area inside the snowflake curve is then

....................$+T$, remembering that T is the area of the original big triangle. (*)

Concentrate on the snowflake curve 'above' the top side of the big triangle. The part of this over the first third of this side—roughly the part inside the oval in the figure—is simply a scaled-down version. The area of this part inside the oval will be

..........................of A. There are four of these scaled-down versions, plus a triangle whose area is

..........................of T making up the area A. So

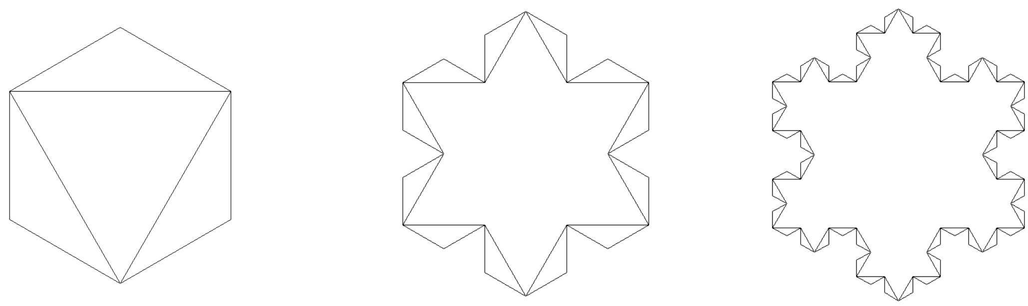
.......................$A+$......................$T = A$.

Solving this for A gives $A =$...................T.

Finally, using the equation (*) above the area inside the snowflake is ....T.

Hexagon construction of a snowflake

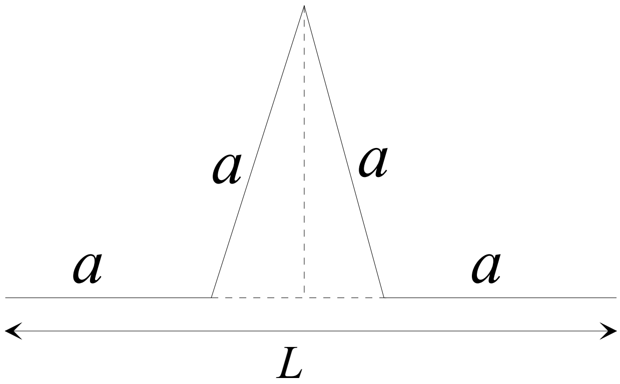
You might like to investigate for yourself the following construction of a snowflake by biting parts out of a hexagon. The figure shows the first three steps, with the triangle construction superimposed. This might suggest to you that the eventual snowflake formed by biting parts out of a hexagon is actually the same as that obtained by adding parts onto a triangle.



If you can carry out the same kind of calculations as those above for this case please send your work to me, Peter Giblin, Department of Mathematical Sciences, The University of Liverpool, Liverpool L69 3BX. There will be prizes for the best efforts.

Some variations on the construction

Instead of taking out the middle third of an edge and replacing it by two sides of an equilateral triangle, as in the above construction of a snowflake, we can replace part of the edge by a two sides of an isosceles triangle of any shape. In the figure, the isosceles triangle has base angles θ and equal sides a.



Making all the lengths of the new edges the same, as in the figure, we have (using some trigonometry), $2a(1+\cos\theta) = L$, giving $a = \frac{1}{2}L/(1+\cos\theta)$. If $\theta = \pi/3$ radians, that is 60 degrees, then this gives $a = \frac{1}{3}L$ since $\cos(\pi/3) = \frac{1}{2}$, as with the original snowflake construction when L is the edge length of a triangle and this edge is replaced by four segments of length $\frac{1}{3}L$. The figure below shows $\theta = 4\pi/9 = 80°$, after one step, after 3 steps and finally after 6 steps. Notice that it almost appears that the edges are making a solid region.

What would happen if $\theta = \pi/2 = 90°$?