THE UNIVERSITY
of LIVERPOOL

Department of Mathematical Sciences

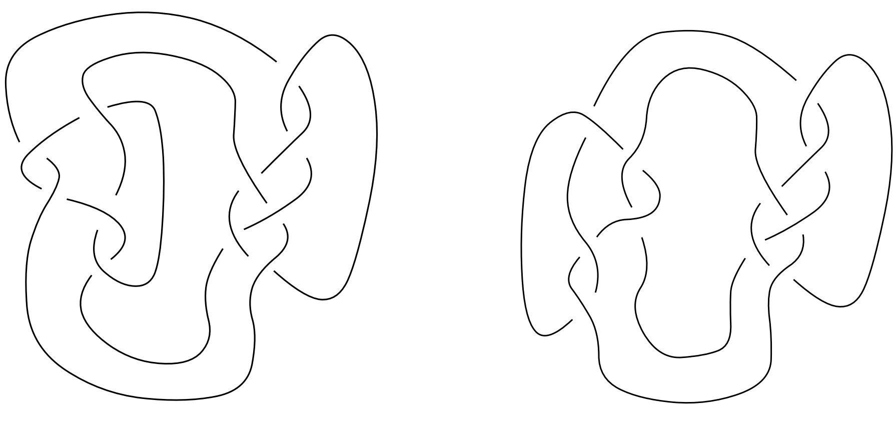
Summer school 2001

Welcome to the Department of Mathematical Sciences!

The first section of this booklet contains a summary of the talk you've just heard. You probably won't need to refer to it, especially as there's a loose **Reference sheet** which lists all the formulae you need to work out the bracket. So you should **start on page 6**.

The rest of the booklet consists of sheets for you to work through. There's more than we expect you to be able to get through today: if you manage to complete the first two sections, you're doing well.

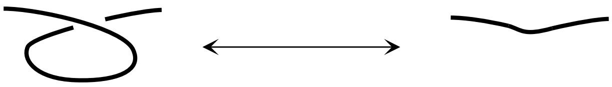
If you have any difficulties, please just ask: that's what the helpers are here for.

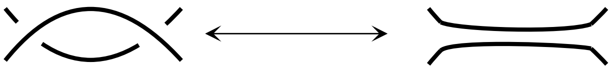


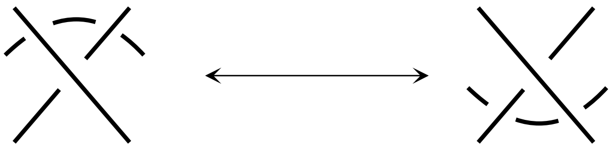
We hope you enjoy this rather different flavour of maths. There are some longer problems which you could try on your own, and there are a few small prizes for the best answers sent in – details inside!

1 Summary

The Reidemeister moves

Move 1: 

Move 2: 

Move 3: 

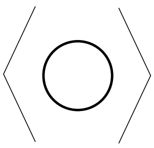
The important thing about the Reidemeister moves is:

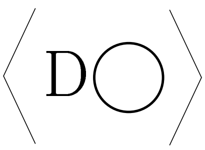
If two diagrams are pictures of the same knot, then you can get from one to the other by repeatedly doing moves 1, 2 and 3.

The aim is to calculate *something* from the diagram of a knot, which gives the same answer however you draw the knot. Then you know that *if two diagrams give different answers, then they're pictures of different knots.*

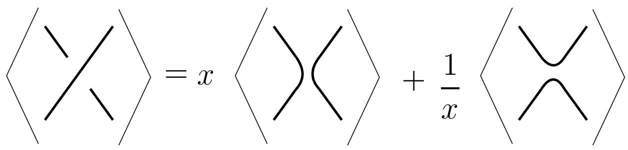
The bracket

Here's a first step towards such a something. The *bracket* of a diagram is given by the following rules:

Rule 1:  $= \mathbf{1}$ *The unknot has bracket* 1

Rule 2:  $= -\left(x^2 + \frac{1}{x^2}\right)\left\langle \mathrm{D} \right\rangle$ D *represents any diagram*

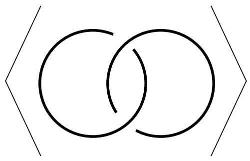
The bracket of a diagram with an extra loop is $-\left(x^2 + \frac{1}{x^2}\right)$ *times the bracket without the loop*

Rule 3: 

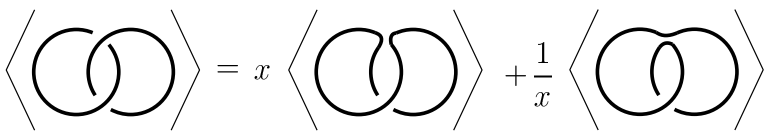
Be sure to get it the right way round. Turning **left** *as you go along the top string gives* x

The idea is that each time you use **Rule 3**, you get one fewer crossing. Once you've applied it enough times, there are no crossings left, and you can use **Rule 2** to work out the answer.

Example: Linked Loops

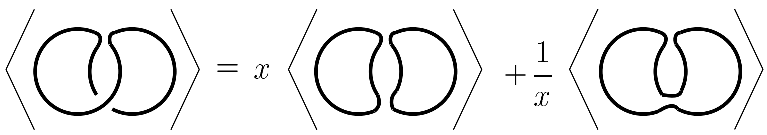
We'll work out  .

First we apply **Rule 3** to the top crossing. We get

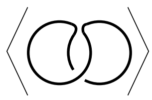


All we've done is to remove the top crossing in each of the two possible ways. When we remove it by turning left along the top string, we multiply by x. When we remove it by turning right, we multiply by $\frac{1}{x}$.

Next, we apply **Rule 3** to the bottom crossing of the first new diagram:

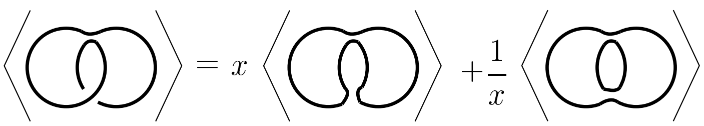


The second picture on the right is just the unknot, so **Rule 1** tells us that this bracket equals 1. The first picture is an unknot together with an extra loop, so **Rule 2** says that this bracket equals $-\left(x^2+\frac{1}{x^2}\right)$. So



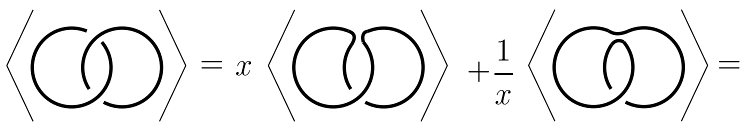
$$= -x\left(x^2+\frac{1}{x^2}\right)+\frac{1}{x} = -x^3-\frac{1}{x}+\frac{1}{x} = -x^3.$$

In just the same way,



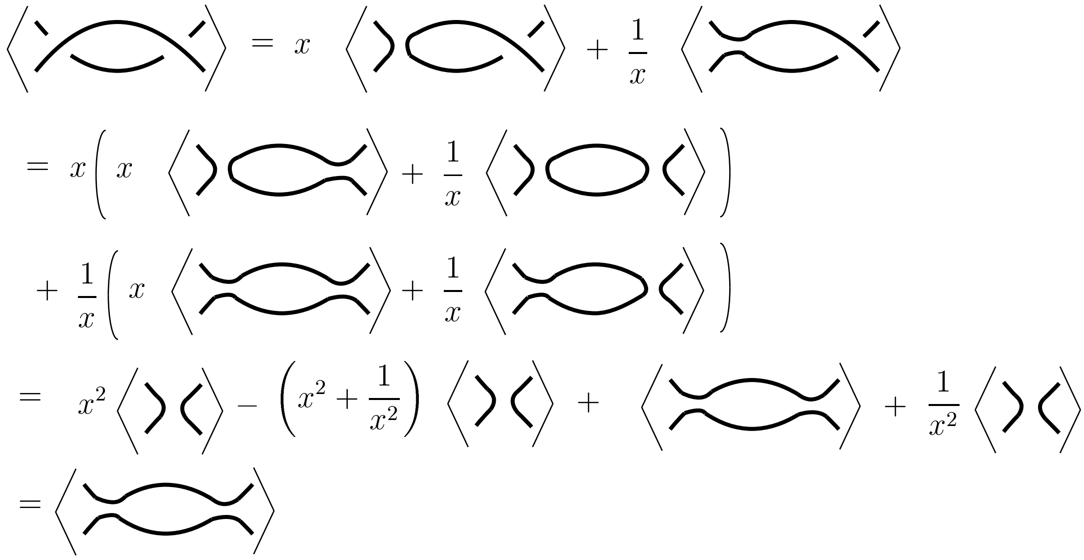
$$= x-\frac{1}{x}\left(x^2+\frac{1}{x^2}\right) = x-x-\frac{1}{x^3} = -\frac{1}{x^3}.$$

So, putting it all together, we get



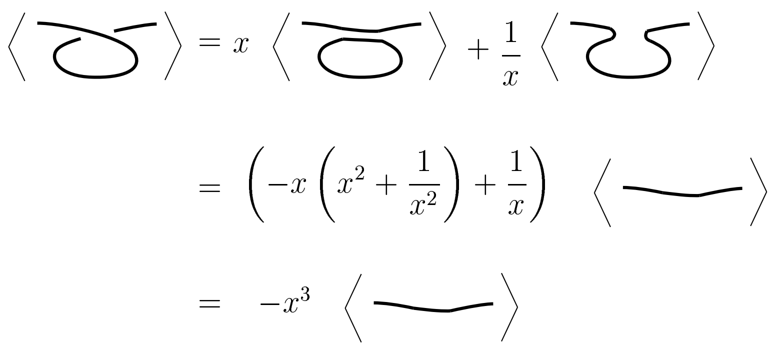
$$= x\left(-x^3\right)+\frac{1}{x}\left(-\frac{1}{x^3}\right) = -x^4-\frac{1}{x^4}$$

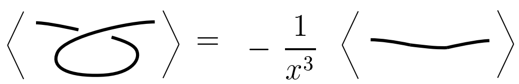
Why should the bracket not change when we draw a knot in a different way? All we need to check is that it doesn't change when we do each of the Reidemeister moves – then it can't change when we do a whole lot of moves taking one picture of a knot to another. Here's how it works for **Move 2**:



So we get the same answer, whether we use the version of the knot with  or the one with . A similar calculation can be done for **Move 3**.

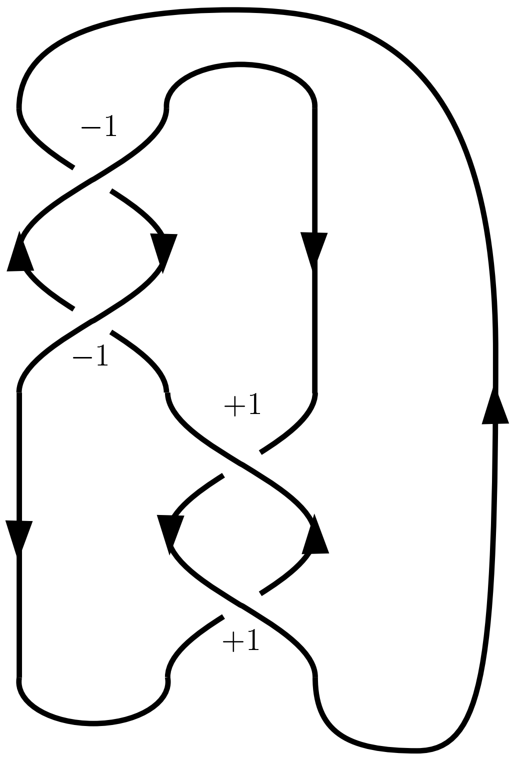
Things go wrong with **Move 1**, though. Here's why:



(and in the same way,  .)

The writhe

Fortunately, this isn't too hard to fix up. Suppose you have a diagram of a knot. Draw arrows all around it in the same direction (if there's more than one loop, you have to choose a direction for each loop).



Now imagine walking over each crossing on the top string, in the direction of the arrow. Look down at someone crossing underneath you in, also following the arrow. If they pass you from *right to left*, as in the lower two crossings, count +1 for that crossing. If they pass you from *left to right*, count −1.

The **writhe** of the knot is the total over all the crossings (so the knot shown above has writhe 0).

Since the writhe of the small loop  is 1, while the writhe of a straight segment $\longrightarrow$ is 0, we can compensate exactly for the problem with **Move 1** by adjusting the bracket to take account of the writhe.

The **square bracket** of a diagram is

$$\left[D \right] = \left(-\frac{1}{x}\right)^{3\times\text{writhe}} \left\langle D \right\rangle$$

(where D represents any diagram). So we work out the bracket and the writhe, and then multiply the bracket by $\left(-\frac{1}{x}\right)^{3\times\text{writhe}}$. The result *doesn't depend on how we've chosen to draw the knot.*

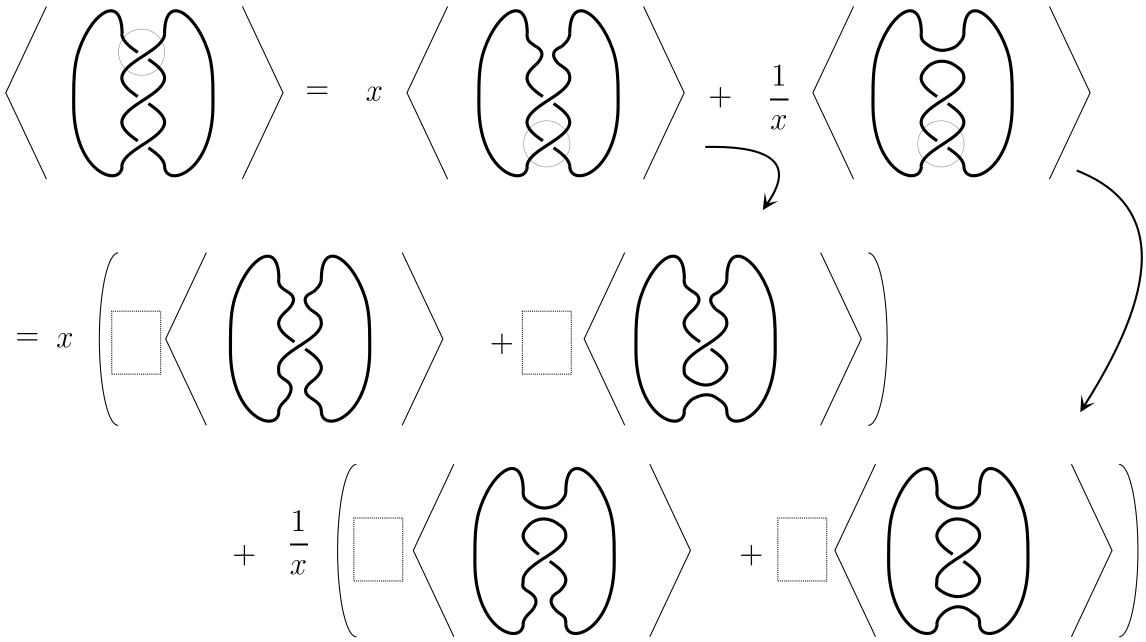
So **If two diagrams have different square brackets, then they are pictures of different knots**.

Since the unknot has square bracket 1, we can also say **if the square bracket of a diagram isn't 1, then it can't be unknotted**.

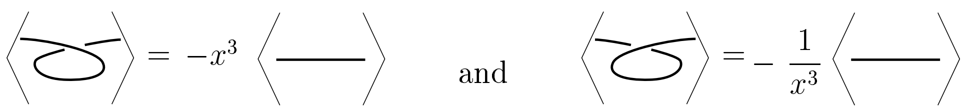
2 Working out a bracket

To be sure that you've understood how to work out the bracket of a knot, try to complete the calculation below. The first step has been done for you. If you have difficulties, try looking at the calculation for the **linked loops** in the summary (page 3). If that doesn't solve your problem, then *ask*: that's what the helpers are here for.

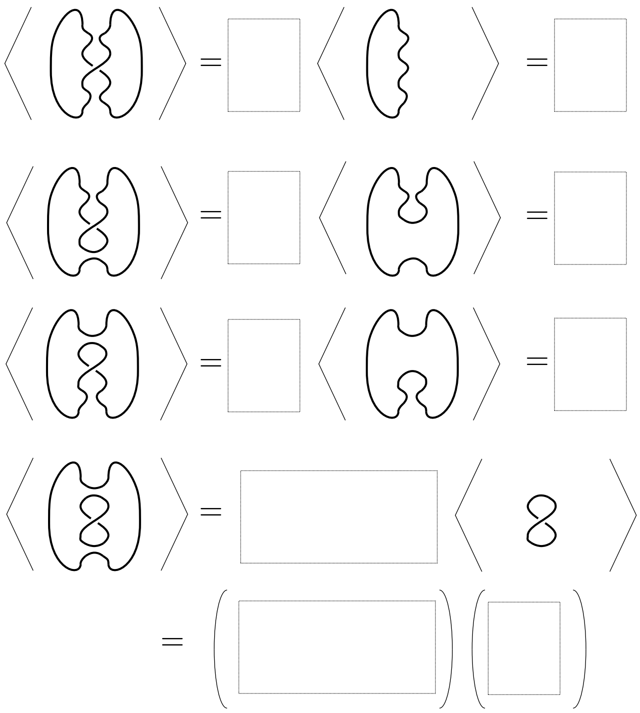
First, fill in the boxes below, using **Rule 3**.



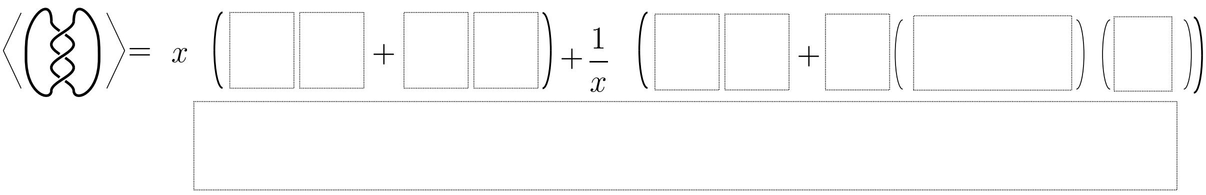
Now you could carry on with the crossings which are left – you'd get 8 diagrams at the end. However, we can work out the 4 diagrams which are left much more quickly using (see page 4)



Apply this, and **Rules 1** and **2** to work out expressions for each of the four brackets which are left – fill in the boxes on the next page. Be sure to get the right version of the rule: when you walk into the small loop along the overcrossing, do you go round clockwise ($-x^3$) or anticlockwise($-\frac{1}{x^3}$)? There are two steps in dealing with the fourth diagram: first use **Rule 2** to get rid of the extra loop, and then treat what's left in the same way as the other three diagrams.



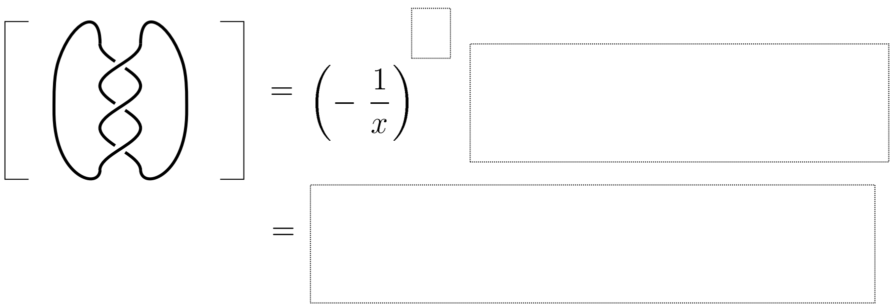
Now put all the bits of your calculation together to work out the bracket: multiply all the terms out as far as you can (you may need some help with this).



To finish off the calculation, you need to work out the *writhe* of the knot. Draw arrows going around the knot (it doesn't matter which direction you choose). Then, for each of the three crossings, imagine walking along the over-crossing in the direction of the arrow. As you look down at someone walking along the under-crossing, do they walk from your right to left (if so count $+1$), or from your left to right (if so count -1)? Add up the results from the three crossings to get

The writhe is

And so... (remember that $\left(-\frac{1}{x}\right)^n$ is the same as $\frac{1}{x^n}$, but with a minus sign if n is odd)



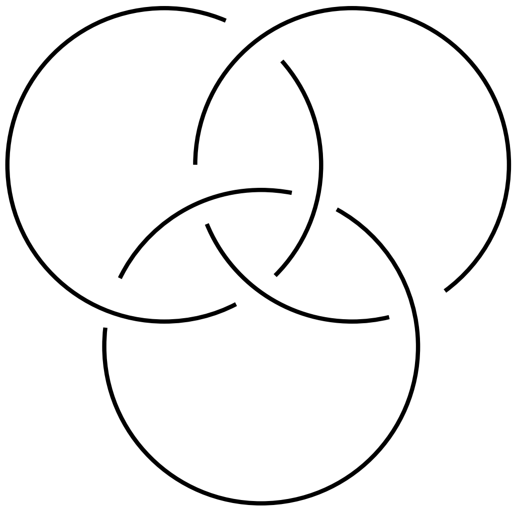
When you've finished, make a real version of your knot out of the string in your folder. Then compare answers with your blue and green friends. Which of the three knots you've worked with are definitely different, and which could be the same? For the two which might be the same, can you move around your pieces of string until you can see that they really are the same?

Could any of these knots be unknotted?

As you can see, working out the bracket takes a long time, even for these simple examples with only three crossings. Imagine how long you'd have to work with the knots on the cover of this booklet! The next two exercises show you a short cut which makes the calculation a little easier.

An example for later

Perhaps you'd like to try going through an example with no help.... Try working out the bracket for the **Borromean rings**,



and show that they really can't be pulled apart. When you work out the writhe, put anti-clockwise arrows around each ring. Don't try it now – it takes a long time! If you manage to complete it, send your solution to

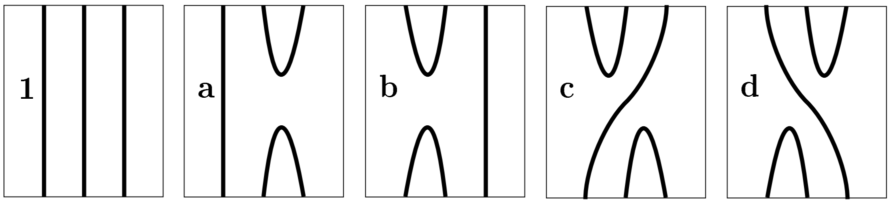
Dr. T. Hall
Department of Mathematical Sciences
University of Liverpool
Liverpool L69 7ZL

The first three correct solutions will win a small prize. Don't forget to include your name and address.

3 An unusual multiplication table

In this exercise, you'll fill out a very unusual multiplication table. The results will be used in the final exercise to get a quicker way of working out the bracket of a knot.

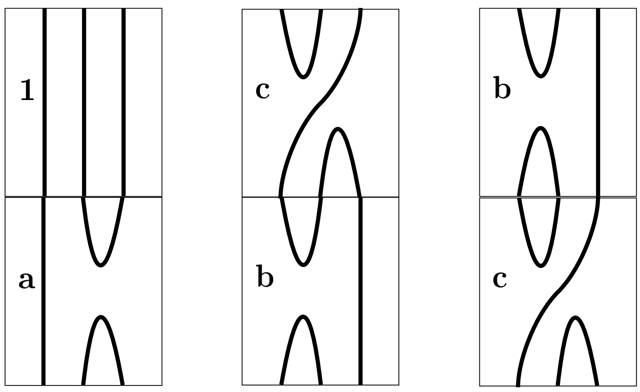
In your folder you should find 5 small rectangles (or 'tiles'), labelled **1**, **a**, **b**, **c**, and **d**.



To multiply two tiles, put the first above the second so that the lines match up: the resulting picture, when you straighten out the lines, should look like a 'stretched-out version' of one of the original tiles. There may be an extra loop floating around. By **Rule 2**, an extra loop means 'multiply by $-(x^2 + \frac{1}{x^2})$': since this will turn up a lot, we'll just use y for it:

$$y = -\left(x^2 + \frac{1}{x^2}\right).$$

Here are some examples:



In the first picture, the two tiles together make a 'stretched out' version of tile **a**. Thus **1** times **a** equals **a**, or **1a** = **a**. (This is why the first tile is called **1**!) In the second picture, straightening out the kink in the middle gives a stretched out version of **b**. Thus **cb** = **b**. The third picture looks like a stretched out version of **c**, but this time with a loop in the middle. Thus **bc** = y**c**.

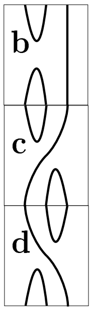
The last two examples show that, unlike with numbers, the order that you multiply matters!

There are 25 possible multiplications you can do with two tiles. Between the group, your task is to complete the multiplication table below: the three examples above have already been filled in. You should work out the shaded entries. Your green and blue colleagues will be doing the rest. Pool your results at the end – they'll be used in the next exercise.

Below

Above	**1**	**a**	**b**	**c**	**d**
1		**a**			
a					
b				y**c**	
c			**b**		
d					

Now suppose you want to work out **bcd**.



You can do this using the multiplication table. First, **bc** $=$ y**c** (where the y means there's a floating loop). So **bc d** $=$ y**c d**, which is just **cd** with an additional floating loop. Look up **cd** on the table, and check that what you get corresponds with what you see when you stack up the tiles **b**, **c**, and **d**.

Try it yourself, to work out **bad1c**, both using the multiplication table and by stacking up the tiles. Either way you do it, you should find that there are two loops, so you get a y^2.

bad1c $=$

Of course, it's easier just to stack the tiles, but if you get lots of the same tiles in a multiplication, you'll have to use the table. Try working out **baba**. You can check your answer by borrowing your neighbour's **a** and **b** tiles.

baba $=$

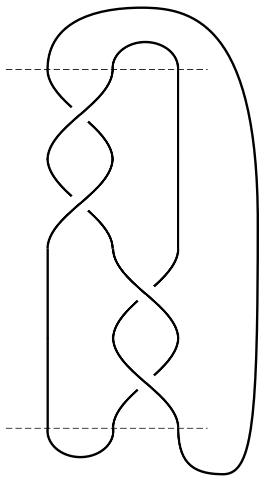
An example for later

What are these five tiles? Take a blank tile, and mark three points on the top edge and three on the bottom edge. Then try to join up the six points with three lines, without any of the lines crossing. There are just five ways of doing it, and that's what the five tiles show.

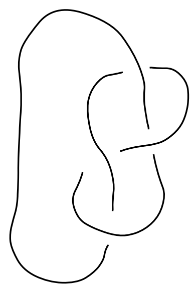
If you put four points on the top edge and four on the bottom edge, then there are fourteen possible patterns. Can you find them all? If so, try to make a multiplication table for them. (You will sometimes get two loops in a picture, so you'll have to put y^2 in the result.)

4 The bracket of the figure 8

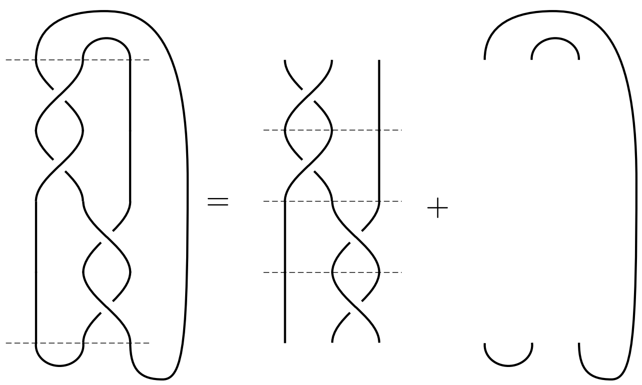
For the final exercise, you're going to use the multiplication table to work out the bracket of a knot with 4 crossings.



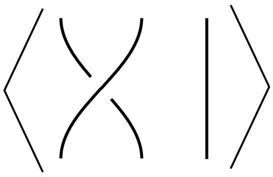
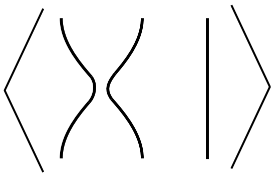
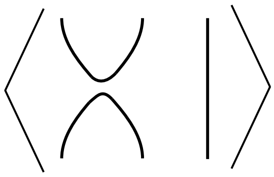
This knot is called the 'figure 8', because it can be drawn more suggestively like this:



We've chosen to draw it in 'braided' form: all of the crossings are contained in a central 'braid' (separated from the rest of the knot by the dotted lines)

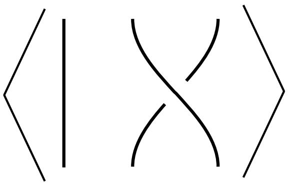


This is often a very useful way to draw knots. In the picture of the braid, the different crossings have been separated by dotted lines. The key point of what we do next is that, if you carry out **Rule 3** on each of these crossings, the resulting pictures will look very like the tiles in the last exercise. Let's try it on the first one:

 $= x$  $+ \frac{1}{x}$ 

$$= x\,\mathbf{1} \; + \; \tfrac{1}{x}\,\mathbf{b}$$

where the $\mathbf{1}$ and the $\mathbf{b}$ represent the tiles from the last sheet. Try it yourself on the third crossing

 $=$

and check that you get the answer $x\mathbf{a} + \frac{1}{x}\mathbf{1}$.

Since the first two crossings and the last two crossings are the same, when we combine them all together we get

$$\left(x\mathbf{1} + \frac{1}{x}\mathbf{b}\right)\left(x\mathbf{1} + \frac{1}{x}\mathbf{b}\right)\left(x\mathbf{a} + \frac{1}{x}\mathbf{1}\right)\left(x\mathbf{a} + \frac{1}{x}\mathbf{1}\right),$$

and all we have to do is multiply out the brackets using the multiplication table. You have to be even more careful than you usually are when multiplying out brackets, because the *order of multiplication matters*. Here's the first two brackets multiplied out:

$$\begin{aligned}\left(x\mathbf{1} + \frac{1}{x}\mathbf{b}\right)\left(x\mathbf{1} + \frac{1}{x}\mathbf{b}\right) &= x^2\mathbf{1} + \mathbf{b} + \mathbf{b} + \frac{1}{x^2}\mathbf{bb} \\ &= x^2\mathbf{1} + 2\mathbf{b} + \frac{y}{x^2}\mathbf{b},\end{aligned}$$

where in the last line we've used (from the multiplication table) that $\mathbf{bb} = y\mathbf{b}$.

Now try it yourself for the third and fourth brackets

$$\begin{aligned}\left(x\mathbf{a} + \frac{1}{x}\mathbf{1}\right)\left(x\mathbf{a} + \frac{1}{x}\mathbf{1}\right) &= \\ &=\end{aligned}$$

and check you get the answer

$$x^2y\mathbf{a} + 2\mathbf{a} + \frac{1}{x^2}\mathbf{1}.$$

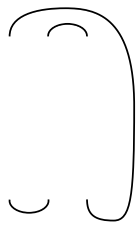
So the whole expression is

$$\left(x^2\mathbf{1} + 2\mathbf{b} + \frac{y}{x^2}\mathbf{b}\right)\left(x^2y\mathbf{a} + 2\mathbf{a} + \frac{1}{x^2}\mathbf{1}\right).$$

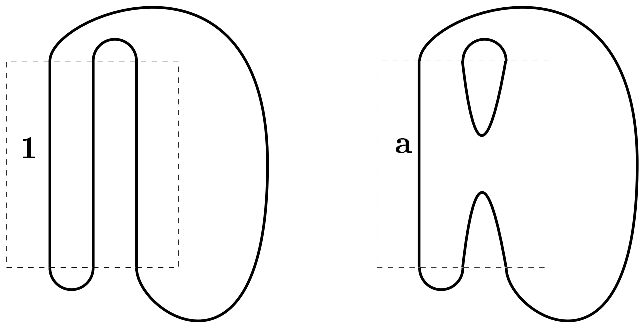
Work this out, using the multiplication table to simplify **ba**: after some work, you should get the answer

$$1 + \left(x^4y + 2x^2\right)\mathbf{a} + \left(\frac{2}{x^2} + \frac{y}{x^4}\right)\mathbf{b} + \left(2x^2y + y^2 + 4 + \frac{2y}{x^2}\right)\mathbf{c}.$$

How do we get the bracket from this? Remember that, in the original picture of the knot, the top and bottoms of the braid were connected like this:



So what **1**, **a**, **b**, and **c** really mean in the above formula is *the bracket of the tile when you join tops to bottoms with these loops.* Since there are no crossings, these brackets are easy to work out. For example, **1** and **a** give



So **a** gives two unlinked loops, which by **Rule 2** has bracket y (i.e. $-\left(x^2 + \frac{1}{x^2}\right)$), and **1** gives the unknot, which by **Rule 1** has bracket 1.

Draw similar diagrams for **b** and **c**, so that you can complete this translation table:

$$\begin{array}{lcl} \mathbf{1} & \longrightarrow & 1 \\ \mathbf{a} & \longrightarrow & y \\ \mathbf{b} & \longrightarrow & \\ \mathbf{c} & \longrightarrow & \end{array}$$

Now, if you're brave, you can work out the bracket of the figure 8 by taking the expression we had earlier

$$1 + \left(x^4y + 2x^2\right)\mathbf{a} + \left(\frac{2}{x^2} + \frac{y}{x^4}\right)\mathbf{b} + \left(2x^2y + y^2 + 4 + \frac{2y}{x^2}\right)\mathbf{c},$$

using the translation table you've just completed, multiplying out all the brackets, and remembering that

$$y = -\left(x^2 + \frac{1}{x^2}\right).$$

If you get

$$x^8 - x^4 + 1 - \frac{1}{x^4} + \frac{1}{x^8}$$

then you're pretty good at this. Send in your solution (the working, not just the answer!): the first three correct ones will win a small prize.