SURREAL NUMBERS

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1. Hackenbush

Hackenbush is a game for two players, Black and White. The playing area consists of the ground and a number of lines of each colour eventually attached to the ground (Fig. 1). The players take turns to remove one of their own lines. Anything which loses contact with the ground floats off and disappears. The loser is the first player unable to move. Who should win the game in Fig. 1?

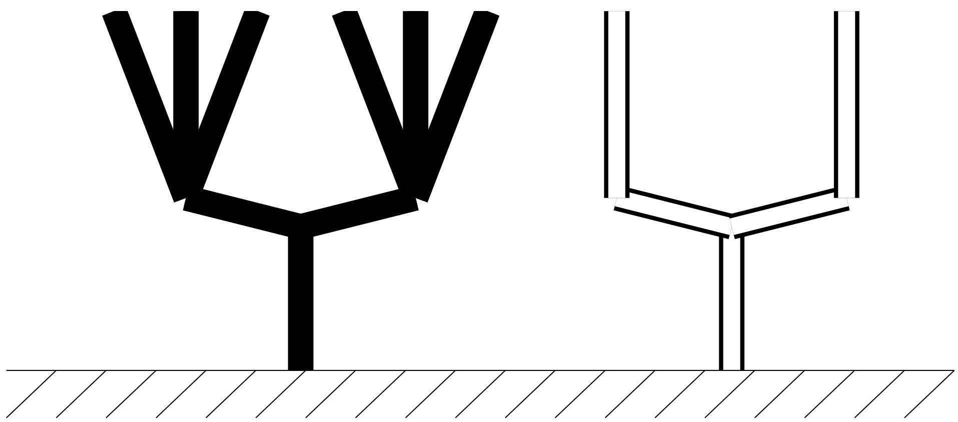

Figure 1

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By counting the lines of each colour, we see that Black has 9 moves and White has 5 moves. So Black has a an advantage of 4 moves. Of course, Black can easily lose his advantage by taking one of the lower branches but with careful play he should win with 4 moves to spare.

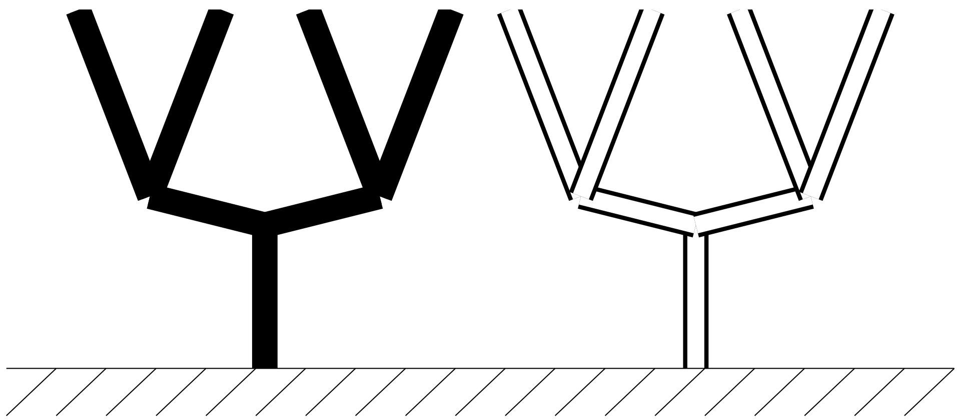

FIGURE 2

What happens when the two sides are evenly matched (Fig. 2)? If White starts, and each player plays carefully, Black will take the last line, so White loses. Similarly, if Black starts, Black will lose. This situation is called a *zero game*. The defining property of the zero game is that the first player to move loses.

It's easy to predict what will happen when the different colours are separated. But when the two different colours appear in the same component, the game becomes more interesting. Who should win the game in Fig. 3? If Black moves first, he must take the lower line, leaving an empty position. White is now unable to move, so she loses. If White moves first, she must take the upper line, leaving the lower. As Black moves next, he must take must take this line, leaving the empty position, so White loses again.

Black certainly has an advantage. How many moves ahead is he? If we give White an extra move (Fig. 4), it turns out that she now has the advantage! (Try analysing the game as we did before). This suggests that Black's original advantage is positive, but less than one move. We can show that in fact Fig. 3 is worth half a move to Black by playing the game in Fig. 5. Try showing that this is a zero game, i.e. that the first player to move always loses. Why does this show that Fig. 3 is worth half a move to Black?

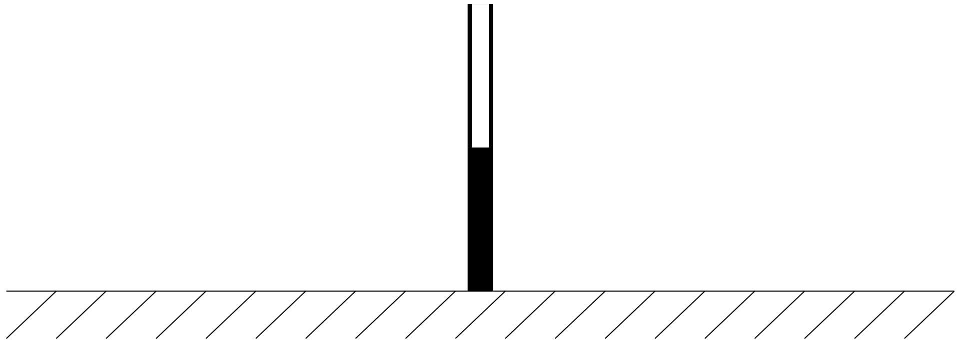

FIGURE 3. $\{0\,|\,1\} = \frac{1}{2}$

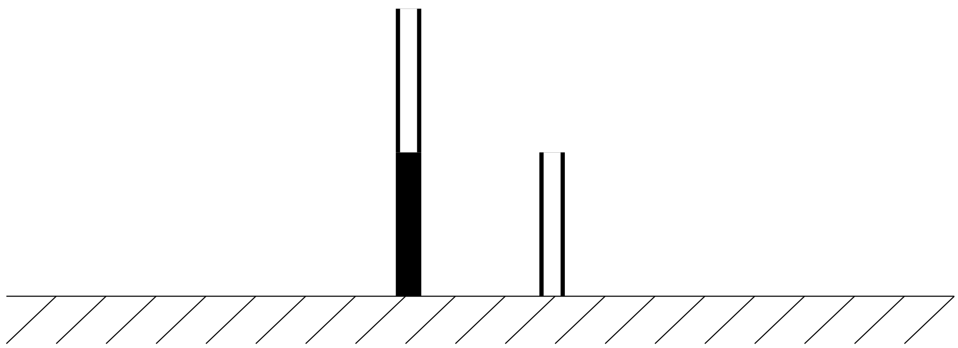

FIGURE 4. $\{0\,|\,1\} + \{\ |\,0\} = -\frac{1}{2}$

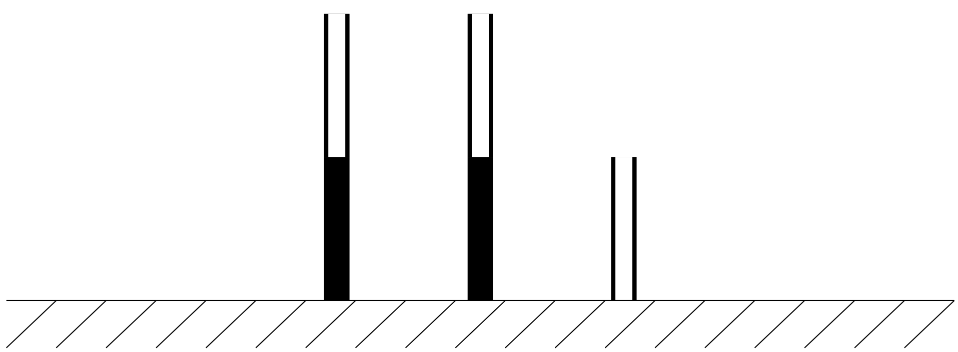

FIGURE 5. $\{0\,|\,1\} + \{0\,|\,1\} + \{\ |\,0\} = 0$

1.1. **Notation.** Conway [1] devised a wonderfully simple notation for analysing games. It's an easy notation to apply to Hackenbush, and it applies to many other games as well. He writes the empty position

(Fig. 6), in which neither player can move, as $\{\ \mid\ \}$. Because this is a zero game (the first player to move loses), we can write

$$\{\ \mid\ \} = 0\,.$$

In Fig. 7, Black can move to the empty position (worth 0), but White

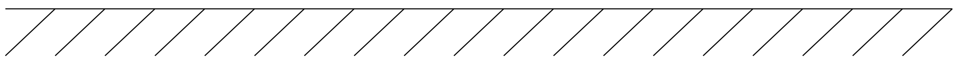

FIGURE 6. $\{\ |\ \} = 0$

cannot move. Black clearly has a one-move advantage. In Conway's notation, we write

$$\{\,0 \mid\ \} = 1\,.$$

Similarly, the position in Fig. 8 is written as

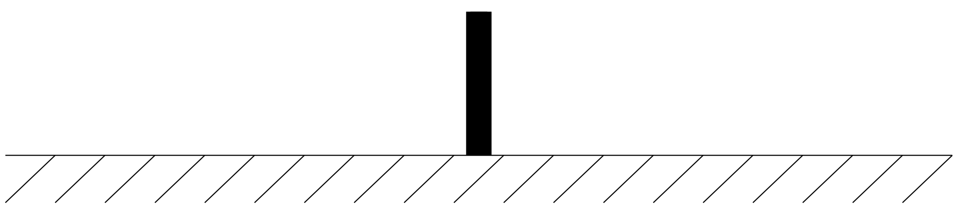

FIGURE 7. $\{\,0\,|\ \} = 1$

$$\{\ \mid 0\,\} = -1\,.$$

The minus sign indicates that White has an advantage over Black. In

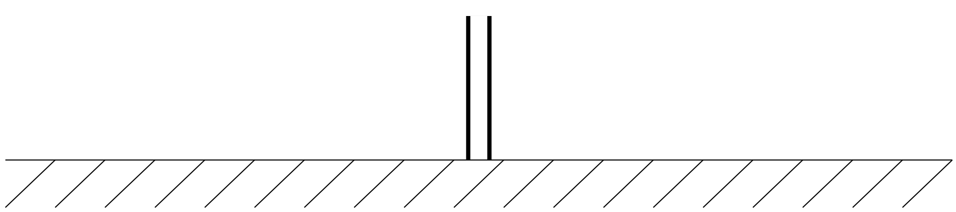

FIGURE 8. $\{\ |\,0\,\} = -1$

the position in Fig. 3, Black can move to the empty position, and White can move to the position in Fig. 7. We showed earlier that Fig. 3 is worth half a move to Black, so in Conway's notation we write

$$\{\, 0 \mid 1 \,\} = \frac{1}{2}\,.$$

In Fig. 5, Black has two options, each leading to a position worth $-\frac{1}{2}$. White has three options. If she takes the lone line, she leaves a position worth 1. If she takes either of the other two lines, she leaves a position worth $\frac{1}{2}$. We already knew that Fig. 5 is worth 0 overall, so we write

$$\left\{ -\frac{1}{2}, -\frac{1}{2} \;\middle|\; \frac{1}{2}, \frac{1}{2}, 1 \right\} = 0\,.$$

Conway's notation for a general position is $\{\, a, b, c, \dots \mid p, q, r, \dots \,\}$. Here, $a, b, c, \dots$ are the values of Black's options, and $p, q, r, \dots$ are the values of White's options.

2. Numbers

Before we go on, we need to reflect on what we have learnt.

1. The strength of a position alone determines the outcome of the game, assuming careful play on each side. Therefore, since different positions may have the same strength, any position can be substituted for any other position with the same strength.
2. Some positions can be separated into several simpler components (e.g. Figs. 4, 5). The crucial point here is that moving in one of the components never affects the state of play in any of the other components. In this case the strength of the position is the sum of the strengths of the components.
3. We can test the strength of a new position by adding a position whose strength we already know and playing the resulting game.
4. Positions with positive strength will be won by Black, positions with negative strength by White, and positions with zero strength by the second player (assuming careful play).

Have you ever thought about what we mean by the word 'number'? In what follows, we will show how Conway managed to define numbers using games. To appreciate this, we need to get used to playing games using only Conway's notation.

For example, I claim that the game $\{\, \frac{1}{2} \mid \;\}$ is worth one move to Black. We can test this by playing the game

$$\left\{ \frac{1}{2} \;\middle|\; \right\} + \{\;\mid 0 \,\}\,, \tag{1}$$

to see if it is a zero game.

If Black moves first, his only option is to move to $\frac{1}{2}$ in the left-hand game. Now White must move from the position

$$\{\, 0 \mid 1 \,\} + \{\ \mid 0 \,\} \,.$$

White has two options, namely to move to 1 in the left-hand game, or two 0 in the right-hand game. The best option turns out to be to move to 1 in the left-hand game. For if White moves to 0 in the right-hand game, the position will be $\{\, 0 \mid 1 \,\} + \{\ \mid\ \}$ with Black to move. He can only move to $\{\ \mid\ \} + \{\ \mid\ \}$, which means that White loses as she is unable to move. But if White moves to $\{\, 0 \mid\ \} + \{\ \mid 0 \,\}$, we now have the sum of a game and its negative, i.e. a zero game. Convince yourself that the player about to move (Black) will lose. So the best option for White is indeed to move to 1 in the left-hand game.

Another way of seeing this is as follows: We know that the left-hand position is worth $\frac{1}{2}$, and the right-hand position is worth -1. If White moves to 1 in the left-hand game, she replaces a position worth $\frac{1}{2}$ to Black by a position worth 1 to Black, so White loses half a move. This is better than losing a whole move by moving from -1 to 0 in the right-hand game. So the players' best moves minimise their losses.

Try completing the proof by showing that if White moves first from position (1), she will lose.

For more practice on playing games using Conway's notation, try showing that $\{\, 1 \mid 2 \,\} = 1\frac{1}{2}$ and $\left\{\, 0 \mid \frac{1}{2} \,\right\} = \frac{1}{4}$. Try adapting the zero game in Fig. 5, which we used to show that $\{\, 0 \mid 1 \,\} = \frac{1}{2}$, to show that $\{\, n \mid n+1 \,\} = n + \frac{1}{2}$, and that $\{\, 0 \mid 2^{-n} \,\} = 2^{-n-1}$. (Here, as earlier, n is a positive integer). Try sketching the corresponding Hackenbush positions.

2.1. **Building up a system of numbers.** The first number to be created is 0. Conway makes 0 from the empty game: $\{\ \mid\ \} = 0$. The next two numbers are $\{\ \mid 0 \,\} = -1$ and $\{\, 0 \mid\ \} = 1$. Continuing, we can make new numbers $\{\, a, b, c, \dots \mid p, q, r, \dots \}$ out of previously created numbers. The numbers $a, b, c, \dots, p, q, r, \dots$ can be any previously created numbers so long as all of $a, b, c, \dots$ are strictly less than all of $p, q, r, \dots$. Even with this restriction, many choices will just be other representations of a previous number. For example, we have seen that $\left\{\, \frac{1}{2} \mid\ \right\} = 1$. It turns out that the new numbers are between the numbers created previously (Fig. 9).

If you continue Fig. 9, you will find that all the fractions have denominators $2, 4, 8, \dots$, i.e. the powers of 2. To get other fractions, and irrational numbers such as π, you need to allow infinite sets on the left

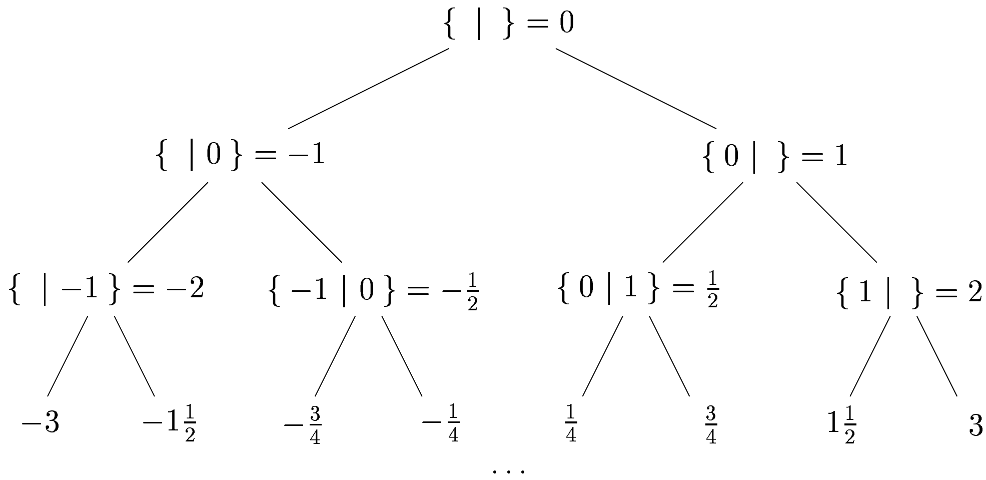

FIGURE 9. A tree of the simplest numbers.

and right. For example,

$$\left\{ \frac{1}{4}, \frac{5}{16}, \frac{21}{64}, \dots \,\middle|\, \frac{3}{8}, \frac{11}{32}, \frac{43}{128}, \dots \right\} = \frac{1}{3},$$

where the left-hand dots stand for all other numbers of the form

$$\frac{1 + 2^2 + 2^4 + \cdots + 2^{2n-2}}{2^{2n}},$$

and the right-hand dots stand for all other numbers of the form

$$\frac{1 + 2 + 2^3 + 2^5 + \cdots + 2^{2n-3}}{2^{2n-1}}.$$

This may seem clumsy, but it is no more so than the decimal expansion $\frac{1}{3} = 0.\dot{3}$ (actually it is the binary expansion $\frac{1}{3} = 0.\dot{0}\dot{1}$ in disguise). And allowing infinite sets introduces wonderful numbers such as the infinite number

$$\omega = \{\, 1, 2, 3, \dots \mid \ \}$$

and its square root

$$\sqrt{\omega} = \left\{ 1, 2, 3, \dots \,\middle|\, \frac{\omega}{1}, \frac{\omega}{2}, \frac{\omega}{3}, \dots \right\}.$$

There are also infinitesimal numbers such as

$$\left\{ 0 \,\middle|\, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots \right\} = \frac{1}{\omega},$$

which is positive, but smaller than any of our usual decimal numbers. The full theory is very beautiful. See any of [1, 3, 5, 6] for more detail.

3. History

I've heard Conway himself speak about his theory of numbers and games. He said he developed his theory of numbers first, after becoming interested in Cantor's theory [4, Ch. 10] of infinite numbers. Later, he tried to analyse the strategies of games such as Go, and realized that he could break complicated games down into sums of simpler games. This lead him to notice that his theory of numbers was just what was needed to understand games.

Conway's theory of numbers became widely known when Knuth's novel [6] was published. It was Knuth who introduced the name 'Surreal Numbers'. Berlekamp [1, p. 79] found an application of the theory, using Hackenbush positions to represent numbers on computers. The Hackenbush representation has advantages over the standard fixed and floating point representations. In Fig. 9, the numbers near the centre are more tightly spaced than the numbers near the edges. This often happens in sets of data which occur in practice, and this is why the Hackenbush system can be advantageous.

References

[1] E. R. Berlekamp, J. H. Conway, and R. K. Guy. *Winning ways for your mathematical plays*, volume 1. Academic Press, London, 1982.

[2] Nancy Casey. Games that interest John Conway. `http://www.cs.uidaho.edu/~casey931/conway/games.html` .

[3] Michael Conrady and David Joyner. Surreal Numbers. `http://www.nadn.navy.mil/mathdept/wdj/surreal1.htm` .

[4] J. H. Conway and R. K. Guy. *The Book of Numbers*. Copernicus, New York, 1996.

[5] Martin Gardner. Conway's Surreal Numbers. In *Penrose Tiles to Trapdoor Ciphers*, pages 49–62. W. H. Freeman, 1989.

[6] D. E. Knuth. *Surreal Numbers: a mathematical novelette*. Addison-Wesley, Reading, Mass., 1974.

[7] Ivars Peterson. Computing in a Surreal Realm. In Ivars Peterson's Mathland, `http://www.maa.org/mathland/mathland_3_18.html`, 18th March 1996.

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