Liverpool University Maths Club, 28 April 2001: Constructions with circles

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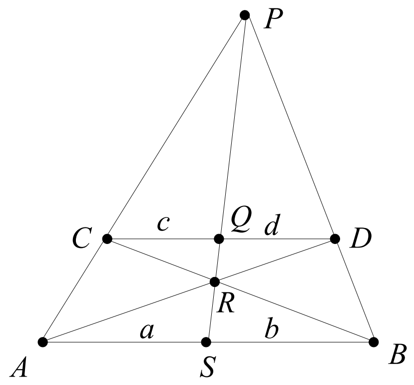
There is a beautiful method of drawing parallel lines which is based on something we did a month ago. In the figure, let us *assume* CD and AB are parallel. Then there are many



pairs of similar triangles in the figure, e.g. PCQ, PAS and PDQ, PBS. From these two pairs we get

$$\frac{c}{a} = \frac{PQ}{PS} = \frac{d}{b}.$$

Similarly from the pairs QCR, SBR and QDR, SAR we get

$$\frac{c}{b} = \frac{QR}{RS} = \frac{d}{a}.$$

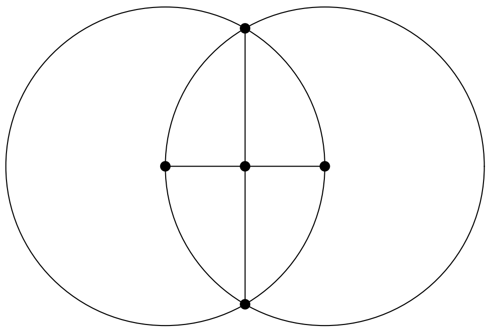
Combining the results $bc = ad$ and $ac = bd$, e.g. by dividing one of these equations by the other, we get $b/a = a/b$ which gives $a^2 = b^2$ and hence $a = b$ since we are dealing with lengths, which are positive numbers. So S is the midpoint of AB and similarly Q is the midpoint of CD.[1]

We can in fact reverse this result to show that, assuming S is the midpoint of AB and that R is found as the intersection of BC and PS, then D is found as the intersection of AR and PB, the line CD will be parallel to AB. This means that we can construct the line

[1]This is a special case of a truly wonderful result from 'projective geometry' concerned with 'complete quadrilaterals' and 'harmonic cross-ratios' which you might meet one day. Projective geometry is an essential tool in applications of mathematics to Computer Vision—the reconstruction of 3-dimensional scenes from 2-dimensional images.

through C parallel to AB provided we can find the midpoint of AB, and are then allowed to use just a 'straight edge', that is an unmarked ruler, only.[2]

It is not possible to find the midpoint of a segment using only an *unmarked* ruler but we can do it if we have also to hand a *pair of compasses*, by means of which we can draw a circle centred at any point and passing through any other point, or of radius equal to the distance between two points in our figure. The figure shows one way of doing this: draw circles



centred at the two points, passing through the other, and then join up the intersections of the two circles. Note that this also constructs an equilateral triangle—can you see which three points in the figure form such a triangle? There are two choices. Note also that exactly the same construction can be used to construct the perpendicular to a line through a point P. In this case we start with the line and the point: with the compasses draw a circle centred at P; this constructs two points on the line with P as their midpoint; now proceed as before.

As another example, we can construct the angle bisector between two given lines. In the

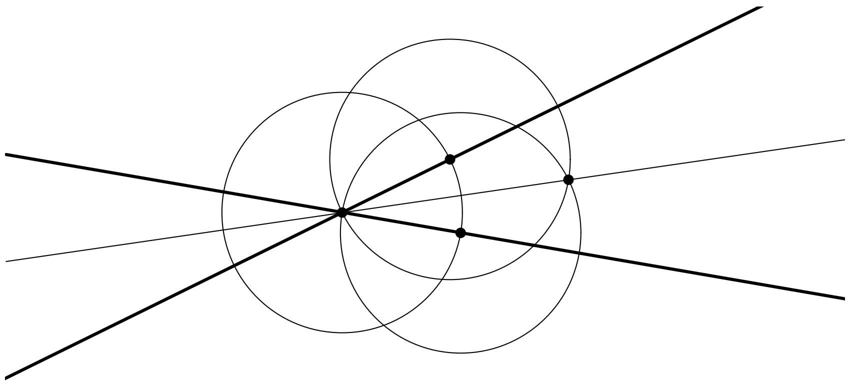
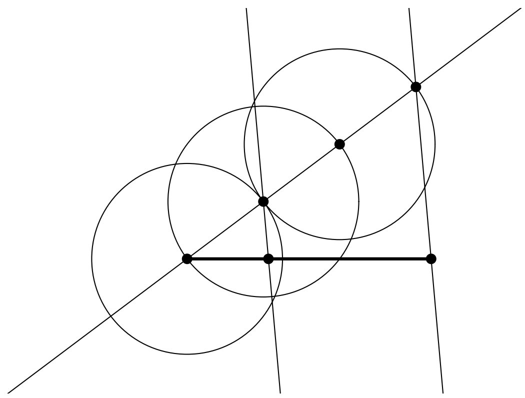


figure the given lines are drawn thickly. Can you reconstruct the steps to draw the angle bisector?

[2]It is not immediately obvious that this 'reversal' of the result proved by similar triangles is true, and if you try a direct argument for it you may find this quite difficult. The proof that the above construction works is usually done by 'proof by contradiction'. This is decidedly subtle and is included here only for completeness. If CD is not parallel to AB then take D' on PB such that CD' is parallel to AB and let AD' meet BC in R'. Then by the result which was proved we have PR' meeting the base AB in the midpoint of AB. But that means that PR' and PR must meet AB in the same point, which is only possible if $R = R'$ and so after all $D = D'$.

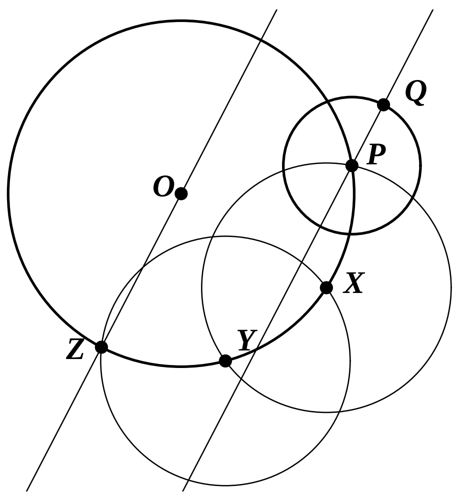
The next figure shows how to divide a segment into three equal parts. It uses the drawing of parallel lines. The segment is divided at a point one-third of the way along its length,



using similar triangles and the fact that the slanting line is divided into three equal parts by the circles construction. Clearly we can also construct the point $\frac{2}{3}$ of the way along the segment by drawing another parallel line.

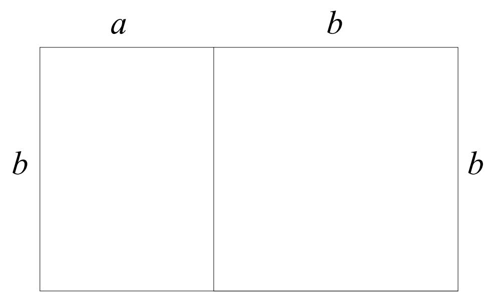
In a similar way a segment can be divided at a point $\frac{p}{q}$ of the way along its length, for any whole numbers p, q.

The next figure is more of a puzzle. The point X is fixed on the circle centre O, and another, smaller circle is drawn centred at a point P on the large circle. Now two more circles and a pair of parallel lines are drawn as shown. Now P is allowed to move round the



circle centre O, carrying the smaller thickly drawn circle with it. This smaller circle always keeps the same radius. Can you predict what will happen to the point Q on the smaller circle? What would happen if, instead of two (thin) circles being drawn before drawing the line through O and parallel line through P, there were more than two?

Here is an entirely different sort of construction problem. A piece of paper is said to have sides in *golden section* if, cutting off a square from the paper as shown, the remaining part is the *same shape* as the original. In the figure, this means that



$$\frac{a+b}{b} = \frac{b}{a}, \quad \text{or} \quad a^2 + ab - b^2 = 0, \quad \text{or} \quad \left(\frac{a}{b}\right)^2 + \frac{a}{b} - 1 = 0.$$

This can be solved (it is a quadratic equation) to give

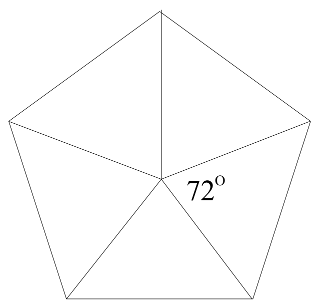
$$\frac{a}{b} = \frac{\sqrt{5}-1}{2},$$

the other root of the quadratic being negative and so not allowable. Now can you construct a rectangle with the golden section for its ratio of sides? (Hint: What is the hypotenuse of a right-angled triangle with sides 1 and 2?) You may assume as known (see above) the method for constructing a line perpendicular to a given one.

Long ago in the autumn in the Maths Club it was shown that

$$\cos\left(\frac{2\pi}{5}\right) = \frac{\sqrt{5}-1}{4}.$$

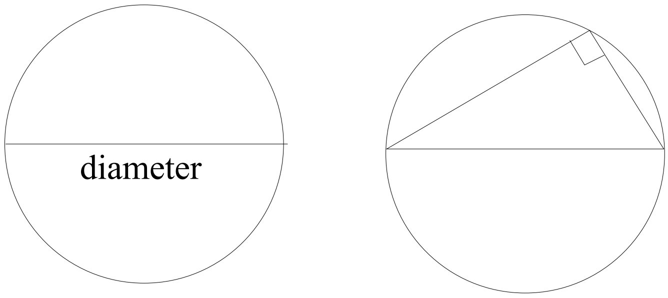
(This is on the webpage, in the 27 October 2000 presentation. Remember the url is http://www.maths.liv.ac.uk/~mathsclub.) Now $2\pi/5 = 72^\circ$ is the angle for a regular pentagon, as in the figure. Can you devise a construction now for a regular pentagon? (The



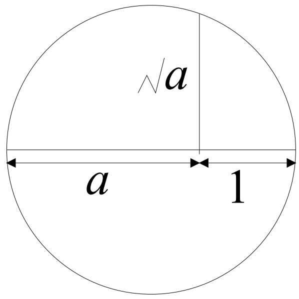
whole thing is to get the angle of 72°. So take a circle and a radius of the circle. You then want to construct another radius of the circle at the angle of 72° to that radius.)

Another challenge of a different kind: Suppose you are given a diameter of a circle, as in the left-hand figure. Your problem is to construct *any* line perpendicular to the diameter, given *only a straight edge*. No compasses allowed this time! (Hint: Remember the 'angle in a semicircle is a right-angle', as in the right-hand figure.)

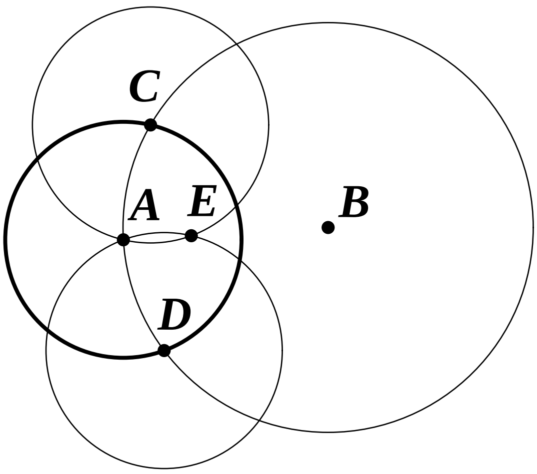
Given a segment of length a (and a 'standard' segment of length 1) construct a segment of length $\sqrt{a}$. The figure contains the necessary hint: given the horizontal line is a diameter



of the circle, can you show that the vertical line has length $\sqrt{a}$? Again this uses the 'angle in a semicircle is a right-angle' result of the previous figure.



In the next figure, one circle, centre A, is fixed. A point B is taken outside this circle,



and then three other circles are drawn as shown, centres B, C and D. This construction produces a point E inside the circle centre A, with AEB a straight line. Can you prove that the product of the lengths AE and AB equals the square of the radius of the fixed circle centre A? We call E the *inverse* point to B with respect to this circle. A harder problem is, given the circle centre A and a point B *inside* it, to construct E on the line through A and B, with $AE \times AB$ equal to the square of the radius.