THE CIRCLE METHOD

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1. League schedules

We will consider the problem that faces the organisers of the World Cup finals: How to construct a five-day league schedule for a group of six teams over so that each team plays each other team once only, and that all six teams are in action on each day.

Day 1	1 v. 2^{1}			3 v. 5^{11}		4 v. 6^{14}
Day 2	1 v. 3^{2}		2 v. 6^{9}		4 v. 5^{13}	
Day 3	1 v. 4^{3}	2 v. 3^{6}				5 v. 6^{15}
Day 4	1 v. 5^{4}	2 v. 4^{7}		3 v. 6^{12}		
Day 5	1 v. 6^{5}	2 v. 5^{8}	3 v. 4^{10}			

Table 1. A fixture list for a league of six teams, obtained by Kirkman's method.

Our first method is due to Kirkman in 1847. First, we list all the fixtures that must take place in *lexicographical order*: 1 v. 2, 1 v. 3, 1 v. 4, 1 v. 5, 1 v. 6, 2 v. 3, 2 v. 4, 2 v. 5, 2 v. 6, 3 v. 4, 3 v. 5, 3 v. 6, 4 v. 5, 4 v. 6, 5 v. 6. The fixtures must take place over five days, with all six teams in action on each day. We must avoid clashes: two fixtures are said to *clash* if the same team takes part in each fixture, e.g. 2 v. 3 clashes with 3 v. 6. And we will take the days in *cyclic order*: we take each day in turn, and after reaching the last day we go back to the first. We start by allocating 1 v. 2 to Day 1. Then we take the next fixture from our list and allocate it to the first Day on which it can be played without clashing. We continue in this way until all the fixtures have been allocated. The final fixture list is shown in Table 1. The fixtures are numbered in the order in which they are allocated.

Our second method is the *circle method*. It is not known who discovered this method. We place team 6 at the centre of a circle, and teams 1–5 around the circumference at the vertices of a regular pentagon. Then we join teams n and $7 - n$ with line segments (Fig. 1). On the first day, the teams at opposite ends of each line segment will

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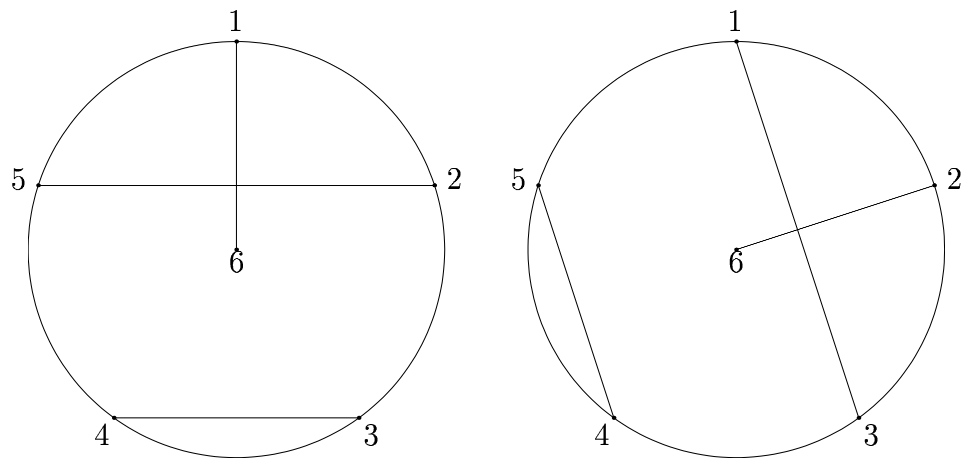

FIGURE 1. The first two day's fixtures, obtained by the circle method.

play each other. On each subsequent day, we rotate the line segments clockwise through a fifth of a revolution, and the teams that are now at opposite ends of line segments will play each other. The German Bundesliga compiles its fixture list using this method.

Remarkably, Kirkman's construction is equivalent to the circle method. Try finding a circle method to generate the fixtures in Table 1!

2. Developments

We have shown how to construct league schedules for any even number of teams. How could we adapt this construction if ...

- The number of teams is odd?
- We require home and away fixtures? Home and away fixtures should alternate as much as possible. What if two of the teams are in the same city, and cannot both play at home on the same day?

The following problems can also be solved by applying a circle method. In some cases you need a point in the centre, and in some you don't.

1. In a seven-storey building the night watchman must patrol the building on the hour between 12am and 6am. He only has time to patrol three floors each hour. Show that he can patrol a different combination of floors each time so that each pair of floors is patrolled together exactly once.
2. A rowing club has thirteen members. Show that thirteen fours can be chosen so that each pair of members rows together only once.

3. Nine young ladies in a school walk out three abreast for four days in succession; it is required to arrange them daily so that no two shall walk abreast twice.
4. Fifteen young ladies in a school walk out three abreast for seven days in succession; it is required to arrange them daily so that no two shall walk abreast twice.
   This problem is due to Kirkman in 1850. Lu Jiaxi c. 1965, and independently Ray-Chaudhuri and Wilson in 1971 proved that the corresponding problem for $6n+3$ young ladies has a solution for any $n \geq 0$.

3. History

Thomas Penyngton Kirkman (1806–1895) spent most of his life as Vicar in the Parish of Southworth, Lancashire. Although he was taught no mathematics at school, he published his first mathematical paper at the age of 40, followed by over 60 works on combinatorics, algebra, geometry and knot theory. In 1857 he was elected a Fellow of the Royal Society, one of the highest honours a scientist can receive.

References

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