Quadrature

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1 Polygons

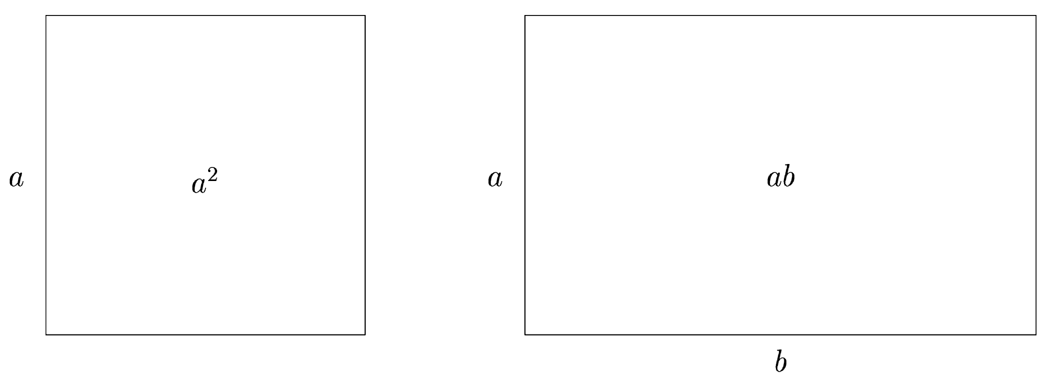

Figure 1: The square and the rectangle.

You are probably familiar with the square and the rectangle (Fig. 1). A square of side a has area a^2, and a rectangle with sides a and b has area ab. The word *quadrature* means the process of determining a square that has an area equal to the area enclosed by a closed curve [2]. This can mean performing a construction of the square with straight edge and compasses, but for us it will mean simply finding the area of a shape (and hence of the square).

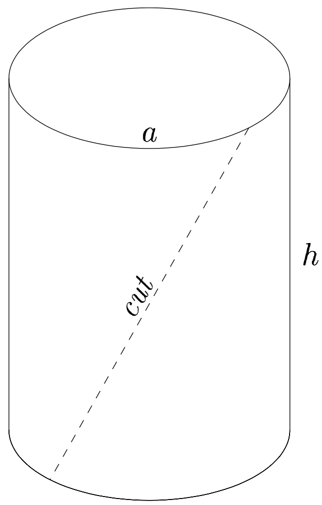

Figure 2: Making a parallelogram from a cylinder.

It is not much harder to find the area of a parallelogram. Take a rectangle with sides a and h, and and make it into a cylinder (Fig. 2). Cut across the cylinder in a straight line at an angle to the rim, and you have a parallelogram. The area of the parallelogram is the same as the area of the original rectangle, ah. There's also a formula involving the two sides of the parallelogram—we need to find the height h in terms of the sides. Let the angle between the two sides be θ (Fig. 3). Drop a perpendicular to the side a to make a right-angled triangle with hypotenuse b and sides a and h. The definition of $\sin\theta$ is

$$\sin\theta = \frac{h}{b},$$

so we have $h = b\sin\theta$. So,

$$\text{area parallelogram} = ab\sin\theta.$$

Cutting the parallelogram in half along one of the diagonals gives

$$\boxed{\text{area } \triangle = \tfrac{1}{2}ab\sin\theta.} \tag{1}$$

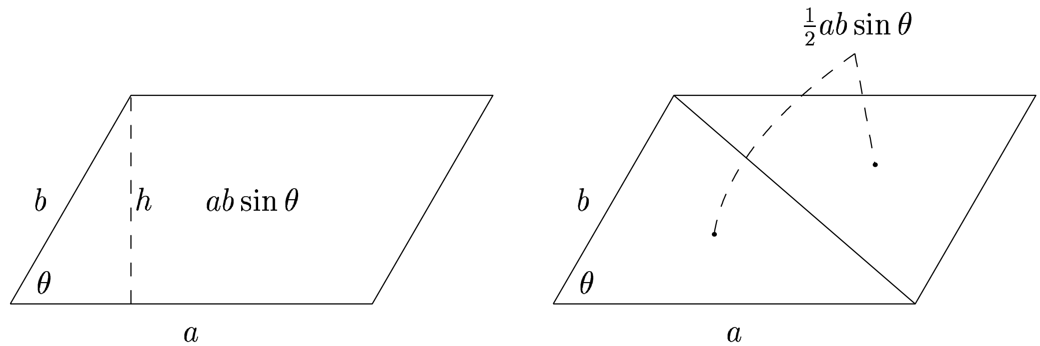

Figure 3: The parallelogram and the triangle.

This is a useful formula for the area of a triangle. You just need to know two sides and the angle between them.

There is a formula for the area of a general quadrilateral which I'm very fond of. Let O be the intersection of the diagonals of the quadrilateral $ABCD$ (Fig. 4). The area $ABCD = \triangle OAB + \triangle OBC + \triangle OCD + \triangle ODA$. Let $OA = a$, $OB = b$, $OC = c$ and $OD = d$. Let $\angle BOA = \theta$. Then $\angle DOC = \theta$ also, and $\angle AOD = \angle COB = 180^o - \theta$. Using Eq. (1) to find the areas, we get

$$\begin{aligned} \text{area } ABCD &= \frac{1}{2}\left(ab\sin\theta + bc\sin(180^o - \theta)\right) \\ &\quad + \frac{1}{2}\left(cd\sin\theta + da\sin(180^o - \theta)\right). \end{aligned}$$

We showed at a previous club that $\sin(180^o - \theta) = \sin\theta$ for any angle θ, so we can simplify:

$$\begin{aligned} \text{area } ABCD &= \frac{1}{2}(ab + bc + cd + da)\sin\theta \\ &= \frac{1}{2}(a + c)(b + d)\sin\theta. \end{aligned}$$

So,

The area of a quadrilateral is half the product of the diagonals multiplied by the sine of the angle between them.

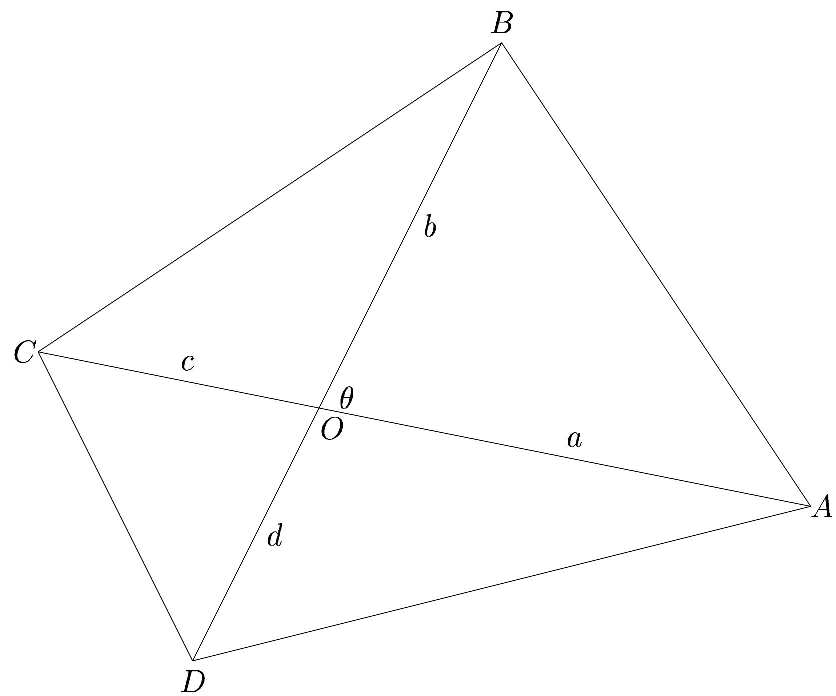

Figure 4: Finding the area of a general quadrilateral.

This isn't boxed because it's a particularly important formula to know, I'm just very fond of it. It's a generalisation of Eq. (1) to quadrilaterals. It's also practical—you can calculate the area of any quadrilateral by measuring the diagonals and the angle between them.

As a further exercise using Eq. (1), try showing that the area A_n of a regular n-sided polygon with side a is [5, p. 201]

$$A_n = \frac{na^2}{4\tan 180^\circ/n}$$

2 Circles

At a previous club we showed that the area of a circle of radius a is πa^2. (Fig. 5). The number π is very special. It is not a fraction. Its decimal expansion goes on and on forever, without repeating. The first hundred decimal places are

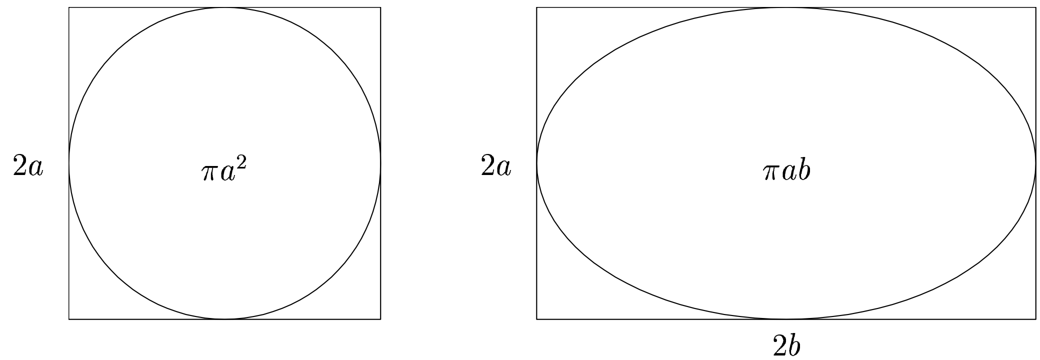

Figure 5: The circle and the ellipse.

$$\pi = 3.14159265358979323846264338327950288419716939937510582097494459230781640628620899 86280348253421170680\ldots$$

Your calculator may also have a decimal approximation.

If we stretch the circle in one direction we get an ellipse (Fig. 5). The surrounding square has become a rectangle, and its area has changed from $4a^2$ to $4ab$, i.e. the area has been multiplied by b/a. The same thing will have happened to the area of the circle, so the area of the ellipse is πab. (The diameter of the ellipse with shortest length is called the minor axis, the diameter with greatest is called the major axis).

Before we do some more work on the circle, I want to introduce a new way of measuring angles. Just as there are lots of ways of measuring length (metres, centimetres, miles, inches, feet, yards, furlongs, chains, leagues, . . .), so there are several ways of measuring angles. You are probably familiar with degrees (360 in a full revolution). My calculator can also deal with grads (400 in a full revolution), and *radians*.

There are $2\pi \approx 6.28\ldots$ radians in a full revolution. Therefore we can work out the adjacent conversion chart. Exercise: How many degrees in one radian? To three significant figures?
Radians may seem a strange unit in comparison to degrees and grads, but they actually simplify many important formulae in mathematics (we'll see some examples soon). When people work with radians, they almost always use multiples of π. For instance, I have no idea where 47 radians is, but I can tell you where 47π radians is straight away.

Degrees	Radians
360	2π
270	$3\pi/2$
180	π
120	$2\pi/3$
90	$\pi/2$
60	$\pi/3$
45	$\pi/4$
30	$\pi/6$

People don't use one unit of length exclusively—you need to be familiar with several. And mathematicians don't use radians exclusively—you'll still need to be familiar with degrees. But we'll be measuring all our angles in radians from now on.

2.1 The sector

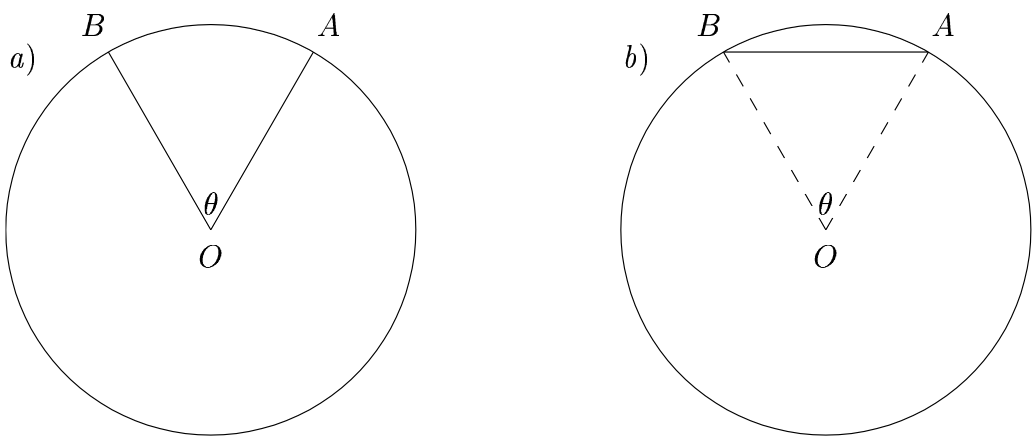

Figure 6: The circular sector and segment.

The sector is a piece of cake (Fig. 6a). More precisely, take two points A, B on the circumference of a circle of radius a, and draw radii OA, OB. Let θ be the angle *in radians* between the radii (we choose to use the smaller angle and calculate the area of the smaller sector, but we could just as well use the reflex angle and calculate the area of the larger sector). If $\theta = \pi$ we would have a semicircle and the area would be $\pi a^2/2$. If $\theta = \pi/2$ we would have an area of $\pi a^2/4$. In general, because there are 2π radians in a full revolution, the fraction of the circle inside the sector is $\theta/2\pi$. So,

$$\boxed{\text{area sector} = \left(\frac{\theta}{2\pi}\right)\pi a^2 = \frac{1}{2}a^2\theta.} \tag{2}$$

Try working out the corresponding formula if θ is measured in degrees.

2.2 The segment

Let A, B be two points on the circumference of a circle centre O with radius a. Again, let θ be the angle *in radians* between the radii OA, OB. The shape enclosed by the chord AB and the arc AB is called a segment (Fig. 6b. Again, we choose the smaller area, but we could just as well choose the larger). The area of the segment is the difference between the area of the sector OAB and the triangle $\triangle OAB$. By Eq. (2), the sector has area $a^2\theta/2$, and by Eq. (1), $\triangle OAB$ has area $a^2 \sin\theta/2$. So,

$$\boxed{\text{area segment} = \tfrac{1}{2}a^2(\theta - \sin\theta).} \tag{3}$$

What is the corresponding formula for θ measured in degrees? If θ is measured in radians again, notice that for $0 \le \theta \le \pi$, the area of the sector OAB is greater than the area $\triangle OAB$. So, in this range of θ, $\theta > \sin\theta$. As θ gets smaller and smaller, the difference in areas gets smaller and smaller, so for θ very small, *and measured in radians*, $\theta \approx \sin\theta$. If your calculator has a radians mode, try using it to calculate $\sin 0.001$. You know it will be less than 0.001, but how much less?

3 More interesting shapes

One of my Maths teachers[1] used to quote the following dialogue [4] :

Spaulding (Groucho Marx) : . . . You say you're going to go to everybody in the house and ask them if they took the painting. Suppose nobody in the house took the painting?

Ravelli (Chico Marx) : Go to the house next door.

Spaulding : Suppose there isn't any house next door?

Ravelli : Well, then of course, we gotta build one.

Spaulding : Well now you're talking. What kind of a house do you think we ought to put up?

He was making the point that you often have to add extra points, lines or other constructions to a diagram in order to solve a problem. Do you need to add anything to the diagrams in this section to make the problems easier to solve?

[1] Mr. Prescott.

3.1 The 50p piece

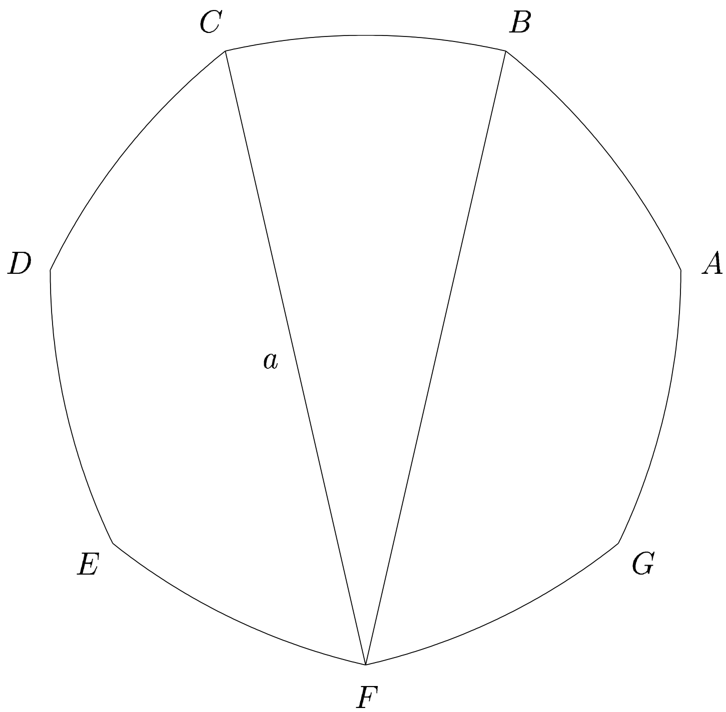

Figure 7: The 50p piece was the world's first seven-sided coin. As we saw in the warm-ups, it has constant diameter, i.e. it can roll between two parallel lines without losing contact with either—useful in slot machines. Let $ABCDEFG$ be a regular heptagon. To form the perimeter of a 50p piece, draw a circular arc AB, centred on the opposite vertex E. Let the radius of the arc be a. Continue, drawing arc BC, centre F, arc CD, centre G, and so on. What is the area of a face of the 50p piece? A solution is given in §5 and [7, q. 5]).

3.2 The lune

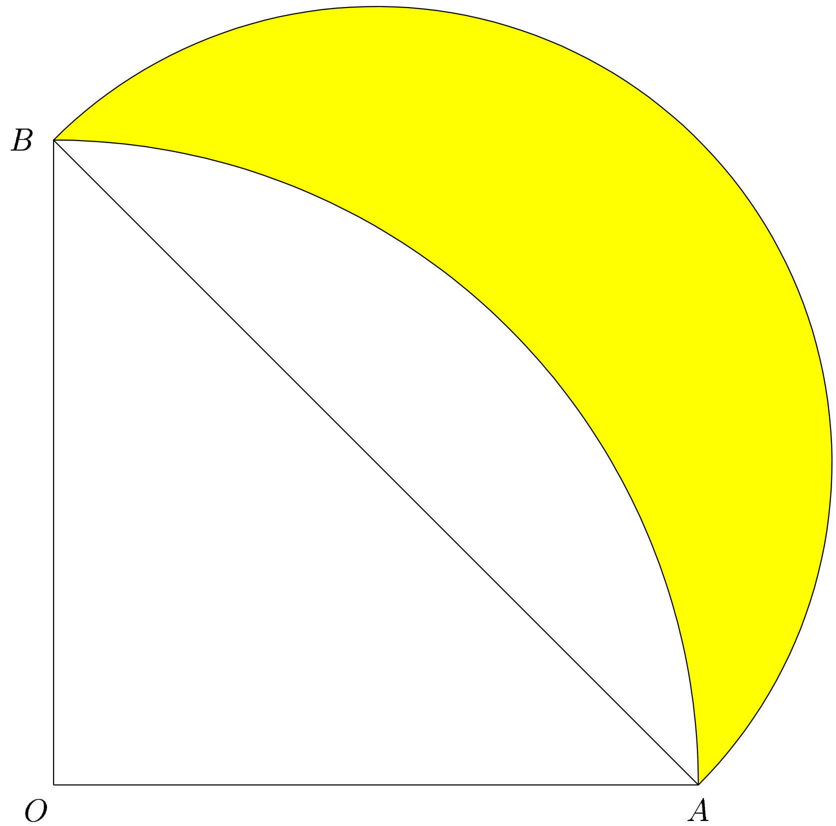

Figure 8: Let $\triangle OAB$ be a right-angled isosceles triangle. Draw an arc AB, center O, and a semicircle with diameter AB. The shape enclosed by the two arcs is called a lune [8, p. 143] and [5, p. 219]. Show that the area of the lune is equal to the area $\triangle OAB$.

3.3 The salinon

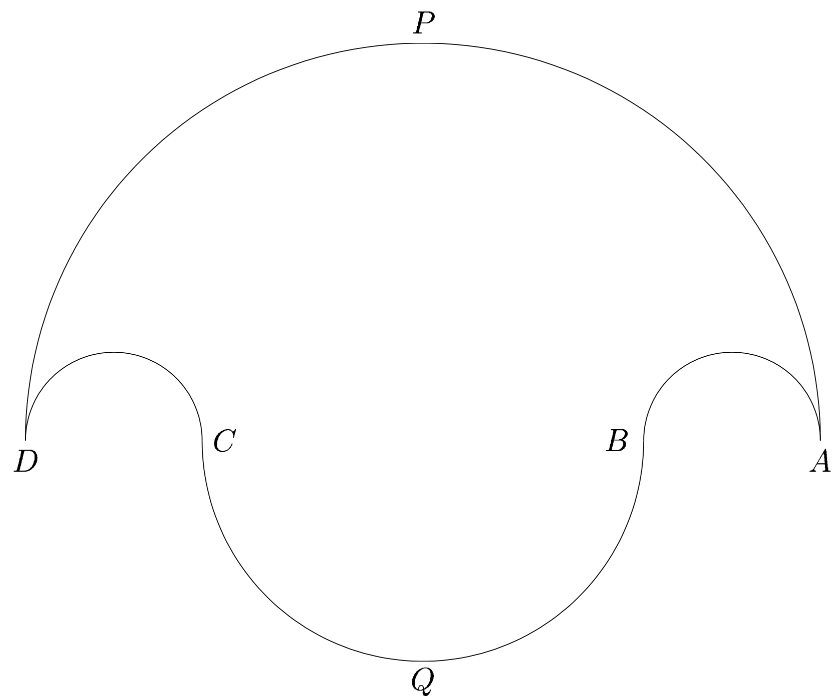

Figure 9: Let $ABCD$ be a line segment with $AB = CD$. Draw a semicircle with diameter AD. Inside this semicircle draw semicircles with diameters AB and CD. Outside, draw a semicircle with diameter BC. The shape enclosed by the four semicircles is called a salinon [8, p. 144]. Let P be the point on semicircle AD furthest from $ABCD$, and Q be the point on semcircle BC furthest from $ABCD$. Show that the area of the salinon is equal to the area of the circle with diameter PQ.

4 History

earlier than 3500 BC Calculation of area of square and rectangle, and triangle as $bh/2$.

c. 1700 BC Recognition of relationship between diameter, perimeter and area of the circle by Babylonians and Chinese

950 BC Quadrature of the circle using $\pi = 3$ (I Kings 7:23).

430 BC Quadrature of the lune by Hippocrates of Chios

300 BC Introduction of degrees by Babylonians

214–212 BC Quadrature of the salinon in Archimedes' *Book of Lemmas.*

190–120 BC Introduction of radians and a quantity equivalent to the sine by Hipparchus of Bithynia.

1116 Quadrature of circular segment by Abraham bar Hiyya of Barcelona, using table of chords and arcs.

1767 Proof that π is irrational by J. H. Lambert

October 1969 Issue of the 30mm diameter 50p piece, the world's first seven-sided coin. The shape is called an 'equilateral curve heptagon.' [6]

1982 Issue of the 20p piece (diameter 21.44mm) [6].

1st September, 1997 Issue of the 27.3mm diameter 50p piece [6].

5 Solution

Because of the rotational symmetry, I think it's a good idea to calculate the area of the curved section OBC (this *isn't* a sector of a circle), and multiply by seven. I know of two methods for calculating this area (there may well be more). First, let's fill in some angles and lengths in Fig. 7 that are used in both methods.

Because $ABCDEFG$ is a *regular* heptagon, so $\angle BOC = 2\pi/7$ (radians), and reflex $\angle BOC = 12\pi/7$. Now consider $\triangle OCF$. Because of the mirror symmetry, line segment OF will bisect reflex $\angle BOC$. So in $\triangle OCF$ (Fig. 10), $\angle FOC = 6\pi/7$. Sides $OC = OF$, because C and F are vertices of a regular heptagon, and O is its centre, hence $\angle OCF = \angle CFO = \pi/14$ (the angles of a triangle must sum to π). Therefore[2] $\angle CFB = 2\angle CFO = \pi/7$.

[2] It is also possible to show that $\angle CFB = \angle COB/2$ by a theorem of circular geometry [3].

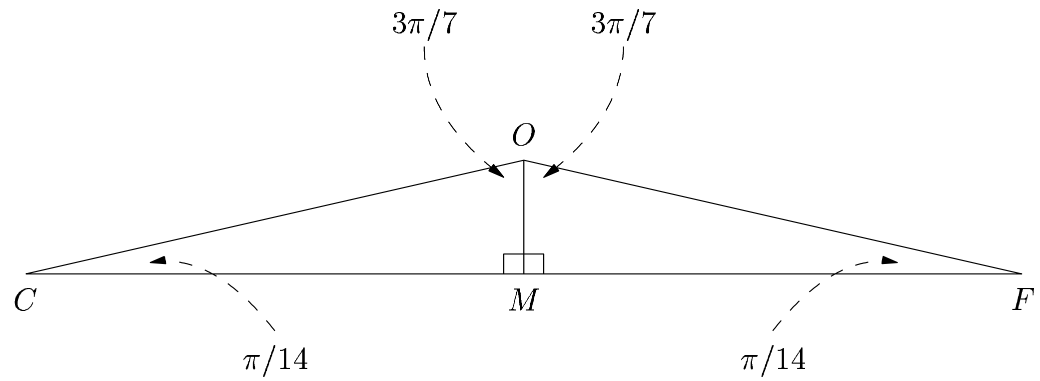

Figure 10: Detail of $\triangle OCF$ in Fig. 7.

Now let M be the midpoint of the side CF, then in right-angled $\triangle CMO$, $\cos(\pi/14) = (a/2)/OC$. So $OC = OF = a/(2\cos(\pi/14))$.

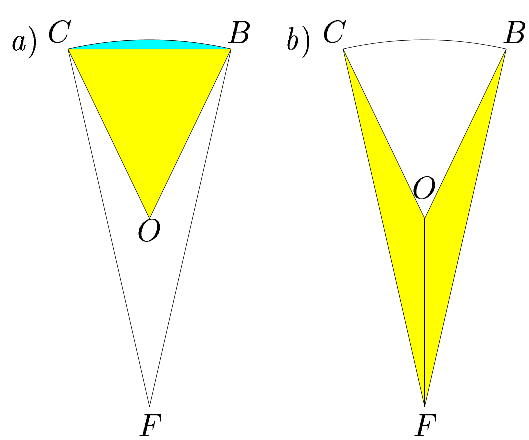

Figure 11: Two methods for finding the area of a face of the 50p piece.

Method 1 Add the area of the circular segment BC and the area of the triangle OBC (see Fig. 11a).

To calcualate the area of the segment BC, remember that F is the centre of the arc BC, *not* O. Then since $\angle CFB = \pi/7$,

$$\text{area segment } BC = \frac{a^2}{2}\left(\frac{\pi}{7} - \sin\frac{\pi}{7}\right)$$

by Eq. (3).

Now for the area of $\triangle COB$. We already know that $OB = OC = 2/(2\cos(\pi/14))$ and $\angle COB = 2\pi/7$, so

$$\text{area } \triangle COB = \frac{1}{2}\frac{a^2}{4\cos^2(\pi/14)}\sin(2\pi/7)$$

by Eq. (1) (the notation $\cos^2\theta$ means $(\cos\theta)^2$).

Therefore

$$\text{area } OBC = \frac{a^2}{2}\left(\frac{\sin 2\pi/7}{4\cos^2\pi/14} + \frac{\pi}{7} - \sin\frac{\pi}{7}\right)$$

Method 2 Calculate the area of the circular sector CFB and subtract the areas of $\triangle FBO$ and $\triangle FOC$ (see Fig. 11b).

We already know $FC = FB = a$ and $\angle CFB = \pi/7$. So

$$\text{area sector } CFB = \frac{a^2}{2}\left(\frac{\pi}{7}\right)$$

by Eq. (2).

Because of the mirror symmetry, $\triangle FBO$ and $\triangle FOC$ are congruent. So we can just find the area of $\triangle FOC$ and double it. We already know all the sides and angles: $OC = OF = a/(2\cos(\pi/14))$, $CF = a$, $\angle OCF = \angle CFO = \pi/14$ and $\angle FOC = 6\pi/7$. Using sides OC and CF, the area is

$$\text{area } \triangle FOC = \frac{1}{2}\,a\,\frac{a}{2\cos\pi/14}\sin\pi/14 = \frac{a^2}{4}\tan\frac{\pi}{14}$$

by Eq. (1). Using sides OC and OF, the area is

$$\text{area } \triangle FOC = \frac{1}{2}\left(\frac{a}{2\cos\pi/14}\right)^2\sin\frac{6\pi}{7}$$

by Eq. (1). These two expressions actually work out to be the same ($\sin 6\pi/7 = \sin \pi/7$ since $\sin(\pi - \theta) = \sin\theta$ for any θ, then you can use the double angle formulae below).

So we have

$$\begin{aligned} \text{area } OBC &= \text{area sector } CFB - 2\text{area } \triangle FOC \\ &= \frac{a^2}{2}\left(\frac{\pi}{7} - \tan\frac{\pi}{14}\right) \end{aligned}$$

Experienced mathematicians know the double angle formulae [1, §10.2]:

$$\begin{aligned} \sin 2\theta &= 2\cos\theta\sin\theta, \\ \cos 2\theta &= \cos^2\theta - \sin^2\theta, \\ &= 2\cos^2\theta - 1, \\ &= 1 - 2\sin^2\theta, \end{aligned}$$

and you can use these to show that both answers are the same (try it—but it's a bit messy). Therefore the area of a face of the 50p piece is

$$\boxed{\frac{a^2}{2}\left(\pi - 7\tan\frac{\pi}{14}\right)}$$

References

[1] Joyce S. Batty. *Pure Mathematics, Book 1.* Schofield & Sims Ltd., Huddersfield, 1986.

[2] John Daintith and R. D. Nelson, editors. *The Penguin Dictionary of Mathematics.* Penguin, 1989.

[3] Euclid. *Elements of Geometry*, Book III, Proposition 20. 'The angle which an arc of a circle subtends at the centre is double the angle which the arc subtends at the circumference'.

[4] *Animal Crackers* (Paramount, 1930). Directed by Victor Heerman. Written by Morrie Ryskind and George S. Kaufman. Story and quotes at `http://www.filmsite.org/anim3.html` .
Audio files at `http://www.whyaduck.com/sounds/wav.htm#Crackers` , including `http://www.whyaduck.com/sounds/crackers/house.wav` .

[5] John Roe. *Elementary Geometry.* Oxford University Press, 1993.

[6] http://www.royalmint.com .

[7] S. T. C. Siklos. Advanced Problems in Mathematics, 1998. Third edition. Copies may be purchased from Publications, OCR, Mill Wharf, Mill Street, Birmingham B6 4BU. Cost £4.50 in 2000.

[8] David Wells. *The Penguin Dictionary of Curious and Interesting Geometry.* Penguin, 1991.