GETTING CLOSE TO THE SQUARE ROOT OF 2. Dr. E.V. Flynn. evflynn@liv.ac.uk
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1. Any nonzero integer can be written as a power of 2 multiplied by an odd number. For example, 44 can be written as $2^2 \times 11$; similarly, $24 = 2^3 \times 3$, $18 = 2^1 \times 9$ and $15 = 2^0 \times 15$. The "power of 2 in n" means the power of 2 you get when you write n in the above form. For example, the power of 2 in 40 is 3, since $40 = 2^3 \times 5$. In general, if $n = 2^r \times$ (an odd number), then r is the power of 2 in n.

What is the power of 2 in each of the following numbers: $128, 55, -160$? What is the power of 2 in each of the square numbers: $1, 4, 9, 16, 25, 36$? For any nonzero integer a, what can you always say about the power of 2 in a^2? Consider the numbers which are twice squares: $2, 8, 18, 32, 50, 72$. For any nonzero integer b, what can you always say about the power of 2 in $2b^2$? Is it ever possible for $a^2 = 2b^2$? Is it ever possible for $\sqrt{2} = \frac{a}{b}$? What about $\sqrt{3}, \sqrt{5}, \ldots$?

2. From question 1, we have seen that it is impossible to find integers a, b for which $\sqrt{2} = \frac{a}{b}$. This is same as saying that $\sqrt{2}$ is never exactly equal to a rational number $\frac{a}{b}$ (i.e. $\sqrt{2}$ is *irrational*); but can we get a good approximation of $\sqrt{2}$ by a rational number $\frac{a}{b}$? That is, can we choose integers a, b so that $\frac{a}{b}$ is very close to $\sqrt{2}$? Let's try to find the best approximation to $\sqrt{2}$ with denominator 11; that is, we want to find which of: $\frac{1}{11}, \frac{2}{11}, \ldots$ is closest to $\sqrt{2}$. We find that $\sqrt{2}$ is between $\frac{15}{11}$ and $\frac{16}{11}$. [*Can you think of a way of discovering this on a calculator, which is faster than looking at all of* $\frac{1}{11}, \frac{2}{11}, \ldots$?]. On a calculator, we see that $\frac{15}{11} - \sqrt{2} = -.050577198$ and $\frac{16}{11} - \sqrt{2} = .040331893$. So, $\frac{16}{11}$ is the best approximation (with denominator 11) to $\sqrt{2}$. Note that the recipricol of .040331893 is 24.79427385, so that $\frac{16}{11}$ is about $\frac{1}{24.79427385}$ away from $\sqrt{2}$. Is this good or bad? Well, the numbers $\frac{1}{11}, \frac{2}{11}, \ldots$ are spaced $\frac{1}{11}$ apart, so that we know in advance that the closest one to $\sqrt{2}$ will be within $\frac{1}{22}$ of $\sqrt{2}$. So, really, being $\frac{1}{24.79427385}$ away from $\sqrt{2}$ is pretty lousy; it's hardly any better than the $\frac{1}{22}$ accuracy we were guaranteed at the outset. In general, amongst fractions with denominator b, namely: $\frac{1}{b}, \frac{2}{b}, \ldots$, we can always find one within $\frac{1}{2b}$ of $\sqrt{2}$, so of course we can always get as close to $\sqrt{2}$ as we like. There will be a fraction with denominator 100 which is within $\frac{1}{200}$ of $\sqrt{2}$, and a fraction with denominator 1000 which is within $\frac{1}{2000}$ of $\sqrt{2}$, and so on.

A *good approximation* to $\sqrt{2}$ is a rational number $\frac{a}{b}$ which is much closer to $\sqrt{2}$ than one would expect with denominator b; that is, which is much closer to $\sqrt{2}$ than $\frac{1}{2b}$. For example, look at fractions with denominator 12. You should find that the closest is $\frac{17}{12}$, and that $\frac{17}{12} - \sqrt{2} = .002453105$, whose recipricol is 407.6466356; that is, $\frac{17}{12}$ is within

$\frac{1}{407}$ of $\sqrt{2}$. That's amazing! It's much better than being within $\frac{1}{24}$. As another way of seeing how close it is, see how close $(\frac{17}{12})^2$ is to 2; this time, using exact fractions. Well, $(\frac{17}{12})^2 - 2 = \frac{289}{144} - 2 = \frac{289-288}{144} = \frac{1}{12^2}$. So, the square of $\frac{17}{12}$ is merely $\frac{1}{12^2}$ away from 2. Let's say that $\frac{a}{b}$ is a *really good* approximation of $\sqrt{2}$ if $(\frac{a}{b})^2$ is at most $\frac{1}{b^2}$ away from 2.

For each denominator b from 1 to 30, find the fraction $\frac{a}{b}$ which is the best approximation to $\sqrt{2}$ with denominator b [for example, $b = 11$ and $b = 12$ have already been done for you above]. In each case, compute $\frac{a}{b} - \sqrt{2}$ (as a decimal), and $(\frac{a}{b})^2 - 2$ (as an exact fraction)? Do not simplify any of your fractions; for example, if $a = 6, b = 4$, write $\frac{a}{b}$ with denominator b, i.e. as $\frac{6}{4}$ (not simplified), and write $(\frac{a}{b})^2 - 2$ with denominator b^2, i.e. as $\frac{4}{4^2}$. Pick out the first few *really good* approximations $\frac{a_1}{b_1}, \frac{a_2}{b_2}, \ldots$. Do you see a pattern? Use the pattern to get the next two. If $\frac{a}{b}$ is not a *really good* approximation, we still say that it is a *pretty good* approximation if $(\frac{a}{b})^2$ is at most $\frac{2}{b^2}$ away from 2. Find the first few *pretty good* approximations. Do you notice a pattern? Use the pattern to get the next two.

Is it ever possible for $(\frac{a}{b})^2$ to be exactly $\frac{3}{b^2}$ away from 2?

3. Consider numbers of the form $a + b\sqrt{2}$, where a, b are integers. These can be multiplied together; for example, $(1+\sqrt{2})(3+2\sqrt{2}) = 1\times 3 + 1\times 2\sqrt{2} + \sqrt{2}\times 3 + \sqrt{2}\times 2\sqrt{2} = 7 + 5\sqrt{2}$. Calculate: $(1+\sqrt{2})(7+5\sqrt{2})$.

We define N by $N(a+b\sqrt{2}) = (a+b\sqrt{2})(a-b\sqrt{2})$. Note that this is the same as $N(a+b\sqrt{2}) = a^2 - 2b^2$ [*explain why*]. For example, $N(4+3\sqrt{2}) = 4^2 - 2\times 3^2 = -2$. Compute $N(3+2\sqrt{2})$. Suppose that $r_1 = a_1 + b_1\sqrt{2}$ and $r_2 = a_2 + b_2\sqrt{2}$. Show that $N(r_1 r_2) = N(r_1)N(r_2)$. Let $r = a + b\sqrt{2}$. Show that $N(r^2) = N(r)^2$, that $N(r^3) = N(r)^3$, and so on.

Let $r = 1 + \sqrt{2}$. What is $N(r)$? Compute $r, r^2, r^3, \ldots$. What is the pattern? Prove this pattern [hint: first expand $(1+\sqrt{2})(a+b\sqrt{2})$]. What do we always know about $N(r), N(r^2), N(r^3), \ldots$? How does this relate to integer solutions x, y of the equation $x^2 - 2y^2 = \pm 1$? How does this relate to question 2? [Hard question for you to think about: how can it be proved that the above sequence gives *all* of the integer solutions to $x^2 - 2y^2 = \pm 1$?]

Let $r = 1 + \sqrt{2}$ and let $s = \sqrt{2}$. What is $N(s)$? Compute $rs, r^2 s, r^3 s, \ldots$. What is the pattern? Prove this pattern. What do we always know about $N(rs), N(r^2 s), N(r^3 s), \ldots$? How does this relate to question 2?

4. Consider: $2,\ 2+\frac{1}{2},\ 2+\frac{1}{2+\frac{1}{2}}, 2+\frac{1}{2+\frac{1}{2+\frac{1}{2}}}, \ldots$, which simplify to: $2,\ \frac{5}{2},\ \frac{12}{5},\ \frac{29}{12}, \ldots$. Compute the next few terms. What are these numbers approaching as a limit?

Consider: $1,\ 1+\frac{1}{2},\ 1+\frac{1}{2+\frac{1}{2}}, 1+\frac{1}{2+\frac{1}{2+\frac{1}{2}}}, \ldots$, which simplify to: $1,\ \frac{3}{2},\ \frac{7}{5},\ \frac{17}{12}, \ldots$. Compute the next few terms. What are these numbers approaching as a limit? Do you recognise the numerators and denominators? Why does this happen? How does this relate to question 2?